Given a discrete dynamical system $\begin{cases} \mathbf{x}_{k+1} = A\mathbf{x}_k & \text{for } k = 0, 1, 2, \ldots \\ \mathbf{x}_0 \text{ given} \end{cases}$, what do the eigenvalues of A tell us?

<u>Ex</u>: Recall the Population Migration Example: $\begin{cases} \mathbf{x}_{k+1} = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix} \mathbf{x}_k & \text{for } k = 0, 1, 2, \ldots \\ \\ \mathbf{x}_0 = \begin{bmatrix} 835,000 \\ 360,000 \end{bmatrix} \end{cases}$

(a). Find the eigenvalues of $A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix}$ and a basis for each eigenspace.

$$|A - \lambda I| = \begin{vmatrix} .94 - \lambda & .02 \\ .06 & .98 - \lambda \end{vmatrix} = (0.94 - \lambda)(.98 - \lambda) - .02(.06) = \boxed{\lambda^2 - 1.92\lambda + 0.92 = 0}$$

(b). Since there are 2 distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0.92$, the associated eigenvectors are

<u>linearly independent and form a basis for $\mathbb{R}^2$</u> . $\Rightarrow$ Write $\mathbf{x}_0$ as a lin. comb. of the eigenvectors.

$$\mathbf{x}_0 = 298750 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 536250 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 835,000 \\ 360,000 \end{bmatrix}$$

(c). Use this eigenvector expansion to compute $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and compare with your answers on p.2.

(d). Use this eigenvector expansion to find a formula for $\mathbf{x}_k$.

(e). Find the limit as $k \to \infty$ of the city and suburb populations (i.e. $t \to \infty$).

Find the solution to the general dynamical system: $\begin{cases} \mathbf{x}_{k+1} = A\mathbf{x}_k & \text{for } k = 0, 1, 2, \ldots \\ \mathbf{x}_0 \text{ given} \end{cases}$

where A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ corresponding to n eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ (not necessarily distinct).

Then since the eigenvectors form a basis for $\mathbb{R}^n$, the initial vector can be written as a linear combination of the eigenvectors:

$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$ where the c_i's can be determined.

Then the solution to the discrete dynamical system is

$$\mathbf{x}_k = c_1(\lambda_1)^k\mathbf{v}_1 + c_2(\lambda_2)^k\mathbf{v}_2 + \ldots + c_n(\lambda_n)^k\mathbf{v}_n$$

Note: By convention (and convenience) label the eigenvalues in order of decreasing magnitude

i.e. $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$

The behavior of the solution can be determined by the magnitude of the eigenvalues:

- $\lambda_i = 1$, then $(\lambda_i)^k \longrightarrow 1$
- $\lambda_i = -1$, then $(\lambda_i)^k \longrightarrow$ oscillates between ± 1
- $|\lambda_i| < 1$, then $(\lambda_i)^k \longrightarrow 0$
- $|\lambda_i| > 1$, then $(\lambda_i)^k \longrightarrow \pm\infty$ (may oscillate)

So the long-term behavior of the solution is determine by the magnitude of the eigenvalues relative to magnitude 1.

Ex: Suppose $\begin{cases} \mathbf{x}_{k+1} = A\mathbf{x}_k & \text{for } k = 0, 1, 2, \ldots \\ \\ \mathbf{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \end{cases}$ where A is a 2×2 matrix

with eigenvalues $\lambda_1 = 1.84$ and $\lambda_2 = 0.43$ and associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(a). Find the general solution and describe what happens to $\mathbf{x}_k$ as $k \to \infty$?

(b). Find the solution to the discrete dynamical system.

(c). Describe the solution as $k \to \infty$.

Population Models (Very simple starting ___linear___ model. More realistic models are ___nonlinear___ .)

Predator-Prey Model: Owls and Rats

Let O_k and R_k represent the number of owls and rats, respectively, in month k.

Assumptions:

- In the absence of rats (primary food source) only half of the owls survive each month.

  i.e. $O_{k+1} = 0.50 O_k$

- In the absence of owls (primary predator), the rat population will grow 10% each month.

  i.e. $R_{k+1} = 1.10 R_k$

- But owls and rats are both present and ___owls eat rats___ :

  - Owls have food (healthy owls) $\Longrightarrow$ population of owls grows ___proportional to the number of rats___ (say 40%)

    i.e. O_{k+1} equation must ___add $0.40 R_k$ term___

  - Rats are killed $\Longrightarrow$ population of rats ___declines proportional to the number of owls___ (say p).

    i.e. R_{k+1} equation must ___subtract pO_k term___

p is called the ___predation parameter___ .

The resulting equations are $\begin{cases} O_{k+1} = 0.50\, O_k + 0.40\, R_k \\ R_{k+1} = -p\, O_k + 1.10\, R_k \end{cases}$ $\quad k = 0, 1, 2, \ldots$

Let $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$ at time k. Rewrite the population model as a matrix equation dynamical system.

$$\begin{cases} \mathbf{x}_{k+1} = \begin{bmatrix} .50 & .40 \\ -p & 1.10 \end{bmatrix} \mathbf{x}_k & \text{for } k = 0, 1, 2, \ldots \\ \mathbf{x}_0 \text{ given} \end{cases}$$

The matrix $A = \begin{bmatrix} .50 & .40 \\ -p & 1.10 \end{bmatrix}$ is the predator-prey matrix.

Ex: Suppose the predation parameter is $p = .081$, then $A = \begin{bmatrix} .50 & .40 \\ -.081 & 1.10 \end{bmatrix}$

(a). Find the general solution to the system.

(b). Describe the behavior as $k \to \infty$ (assume $c_1 > 0$)

Section 5.6, p. 309: #1, 2, 3, 4, 5, 6