<u>THEOREM</u> If A and B are $n \times n$ matrices, which are similar, then they have the same characteristic equation and hence the same eigenvalues.

<u>PROOF</u> Let A and B be similar $n \times n$ matrices. Then

$$
\begin{aligned}
B &= \underline{\quad P^{-1}AP \quad} \text{ for some invertible matrix } P \\
B - \lambda I &= P^{-1}AP - \lambda I \\
&= P^{-1}AP - \lambda \underline{\quad P^{-1}P \quad} \qquad \text{since } I = P^{-1}P \\
&= P^{-1}AP - P^{-1}\lambda P \qquad \text{since scalars } \underline{\quad \text{commute} \quad} \text{ with matrices} \\
&= P^{-1}\underline{\quad (AP - \lambda P) \quad} \qquad \text{by factoring out } P^{-1} \\
&= P^{-1}(A - \lambda I)P \qquad \text{by factoring out } P
\end{aligned}
$$

Take the determinant of both sides:

$$
\begin{aligned}
\det(B - \lambda I) &= \det\left[P^{-1}(A - \lambda I)P\right] \\
&= \underline{\quad \det(P^{-1})\det(A - \lambda I)\det(P) \quad} \qquad \text{by properties of determinants (Theorem 6, Sec 3.2)} \\
&= \det(P^{-1})\det(P)\det(A - \lambda I) \qquad \text{since determinants are } \underline{\quad \text{scalars} \quad}\text{, they commute} \\
&= \underline{\quad \det(P^{-1}P)\det(A - \lambda I) \quad} \qquad \text{by properties of determinants (Theorem 6, Sec 3.2)} \\
&= \det(I)\det(A - \lambda I) \\
&= \underline{\quad \det(A - \lambda I) \quad}
\end{aligned}
$$

i.e. $\det(B - \lambda I) = \det(A - \lambda I)$

Therefore, A and B have the same characteristic polynomial and hence, the same eigenvalues. ■

1. Determine whether the following statement is true or false. <u>If it is true, prove it. If it is false, give a counter-example.</u>

TRUE OR FALSE: If A and B are row equivalent, then they have the same eigenvalues.
[Hint: Consider matrices whose eigenvalues are really easy to find.]

If a matrix A is similar to a matrix with a simple form (e.g. a diagonal matrix), then it can help with many computations.

Ex: Given the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, compute $D^3 = DDD$. Show all your work.

2. Based on the last example, complete the following statement:

Let D be a diagonal $n \times n$ matrix, i.e. $D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & a_{nn} \end{bmatrix}$. Then $D^k = \begin{bmatrix} a_{11}^k & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^k & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & a_{nn}^k \end{bmatrix}$.

THEOREM Let A be an $n \times n$ matrix that is similar to a diagonal matrix D. Then $A^k = PD^kP^{-1}$.
[Proof on next page.]

Ex: Given $A = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, and $P = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}$,

(a). Verify that A and D are similar by showing that $AP = PD$ and verify that P is invertible.

(b). Use the theorem to compute A^5

(c). After finding P^{-1} and easily computing D^5, you only needed to do 2 matrix multiplications ($P①D^5②P^{-1}$) instead of 4 if computing A^5 directly without the theorem ($A①A②A③A④A$).
If you computed A^{100} directly you would perform ___99___ matrix multiplications, but using the theorem you would still only use ___2___ . Can you see the computational advantage? (rhetorical)

PROOF Let A be an $n \times n$ matrix that is similar to a diagonal matrix D.

That is, ___$A = PDP^{-1}$___ for some invertible matrix P. [Show $A^k = PD^kP^{-1}$.]

Basis ($k = 2$):

$$\begin{aligned} A^2 &= AA \\ &= \underline{(PDP^{-1})(PDP^{-1})} \\ &= (PD)(P^{-1}P)(DP^{-1}) \\ &= \underline{(PD)(I)(DP^{-1})} \\ &= (PD)(DP^{-1}) \\ &= \underline{P(DD)P^{-1}} \\ &= PD^2P^{-1} \quad \surd \end{aligned}$$

Induction: Assume true for $k = n$ (i.e ___$A^n = PD^nP^{-1}$___). [Show true for $k = n + 1$.]

$$\begin{aligned} A^{n+1} &= A^nA \\ &= \underline{(PD^nP^{-1})}(PDP^{-1}) \quad \text{by the induction assumption.} \\ &= (PD^n)(P^{-1}P)(DP^{-1}) \\ &= \underline{PD^n(I)DP^{-1}} \\ &= PD^nDP^{-1} \\ &= \underline{PD^{n+1}P^{-1}} \end{aligned}$$

Thus, it is true for $k = n + 1$.

Therefore, by induction it is true for all k ___≥ 2___. [Note: It is true for $k = 1$ by the definition of similarity.] ■

DEF An $n \times n$ matrix A is said to be **diagonalizable** if there exists and invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

But how do we actually diagonalize a matrix A? i.e. How do we find the matrices P and D? (rhetorical)

<u>THEOREM</u> **The Diagonalization Theorem**
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

<u>PROOF</u> Let A be an $n \times n$ matrix.

$\Longrightarrow$: Let A be <u>diagonalizable</u> . [Show that A has n linearly independent eigenvectors.]

Then <u>$A = PDP^{-1}$</u> for a diagonal matrix D and an $n \times n$ matrix P.

$\Rightarrow AP = PD$ by matrix multiplication and simplification.

Since A and D are <u>similar</u> , the they have the same eigenvalues.

Hence the diagonal entries of D are the <u>eigenvalues</u> of A.

i.e. $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the <u>columns</u> of P. That is $P = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n]$.

Then $AP = A[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n] = [$ <u>$A\mathbf{v}_1\ A\mathbf{v}_2\ \ldots\ A\mathbf{v}_n$</u> $]$.

And <u>PD</u> $= [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2\ \ldots\ \lambda_n\mathbf{v}_n]$

Since these two products are equal (i.e. $AP = PD$), we have $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1,\ A\mathbf{v}_2 = \lambda_2\mathbf{v}_2,\ \ldots, A\mathbf{v}_n = \lambda_n\mathbf{v}_n$. $(*)$
[Before we can claim that these vectors are eigenvectors of A, we must show that they are <u>nonzero</u> .]

Since P is <u>invertible</u> , the Invertible Matrix Theorem says that the columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a <u>linearly independent</u> set.

Then all the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are nonzero, otherwise they would be <u>dependent</u> .

Therefore, by $(*)$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are n <u>eigenvectors</u> of A, which are linearly independent .

$\Longleftarrow$: Let A have n linearly independent eigenvectors. [Show that <u>A is diagonalizable.</u>]
<u>**Finish later as homework**</u>.

The previous proof shows us how to find P and D and thus diagonalize A. Complete the following corollary based on your work from the proof of the Diagonalization Theorem.

<u>Corollary</u> A matrix A is similar to a diagonal matrix D (i.e. $A = PDP^{-1}$) if and only if the columns of P are n linearly independent ___eigenvectors___ of A. Furthermore, the diagonal entries of D are the ___eigenvalues___ of A corresponding, ***respectively***, to the eigenvectors in P.

<u>Ex</u>: Given that A is factored into the form PDP^{-1} below, use the corollary above to determine the eigenvalues of A and a basis for each eigenspace without performing any work.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda = 4$ with basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\lambda = -2$ with basis: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Homework: Finish proof, p. 4
Section 5.1, p. 271: #15, 19, 20, 21, 25, 27
Section 5.2, p. 279: #15, 17, 18, 21, 23, 24
Section 5.3, p. 286: #2, 3, 5