<u>Def</u> The **null space** of an $m \times n$ matrix A, denoted Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation

$$\text{Nul } A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

<u>Note</u>: Is $\mathbf{u}$ in the null space of A? IS EQUIVALENT TO ASKING Does $A\mathbf{u} =$ _______?

<u>Ex</u>: Is $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 4 \end{bmatrix}$ in the null space of $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 0 & 1 & 6 & 1 \\ -3 & 4 & 5 & 3 \end{bmatrix}$?

$\boxed{\text{THEOREM The null space of an } m \times n \text{ matrix } A \text{ is a subspace of \_\_\_\_\_\_\_\_\_.}}$

[Hint: Do the $\mathbf{x}$ vectors in Nul A come from $\mathbb{R}^n$ or $\mathbb{R}^m$?]

<u>Proof</u> Let A be an $m \times n$ matrix. [Show that Nul A is a subspace of $\mathbb{R}^n$.]

(0). Let $\mathbf{x} \in$ Nul A. Then $\mathbf{x} \in$ _________ by definition of Nul A. Thus, Nul A is a ____________ of $\mathbb{R}^n$.

(1). $\mathbf{0} \in$ Nul A since ____________ is a solution to _____________ i.e. $A\mathbf{0} = \mathbf{0}$.

(2). Let $\mathbf{u}$ and $\mathbf{v}$ be in Nul A. [Show that $\mathbf{u} + \mathbf{v} \in$ Nul A].

Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. [Show that _________________ .]

$$\begin{aligned} \Rightarrow A(\mathbf{u} + \mathbf{v}) &= A\mathbf{u} + A\mathbf{v} && \text{by multiplication properties} \\ &= \underline{\qquad\qquad} && \text{since } \mathbf{u}, \mathbf{v} \in \text{Nul } A \\ &= \mathbf{0} \end{aligned}$$

i.e. $A(\mathbf{u} + \mathbf{v}) = \mathbf{0}$.

Thus $\mathbf{u} + \mathbf{v} \in$ Nul A. (i.e. Nul A is closed under addition.)

(3). Let c be a scalar. [Show that $c\mathbf{u} \in$ Nul A.]

$$\begin{aligned} \Rightarrow A(c\mathbf{u}) &= \underline{\qquad\qquad} && \text{by multiplication properties} \\ &= c\mathbf{0} && \text{since } \mathbf{u} \in \text{Nul } A \\ &= \mathbf{0} \end{aligned}$$

i.e. $A(c\mathbf{u}) = \mathbf{0}$

Thus $c\mathbf{u} \in$ Nul A. (i.e. Nul A is closed under ________________________ .)

Therefore, Nul A is a subspace of $\mathbb{R}^n$. ■

<u>Ex</u>: Find a spanning set for the null space of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 7 & 3 & -1 & 3 \\ 3 & 1 & -1 & 1 \end{bmatrix}$.

i.e. Find a set of vectors whose span is the set of ________________ to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Step 1. Solve $A\mathbf{x} = \mathbf{0}$ and write the solution(s) in parametric vector form.

$$\text{RREF}(A) \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[Careful. This is just the coefficient matrix.]

Step 2. The vectors found in Step 1 form a spanning set for Nul A. [We'll discuss this more as a class.]

<u>Def</u> The <u>**column space**</u> of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. In other words,

$$\text{Col } A = \_\_\_\_\_\_\_\_\_\_ \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} \qquad \text{where } \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \text{ are the columns of } A.$$

<u>Theorem</u> The column space of an $m \times n$ matrix A is a subspace of __________ . [$\mathbb{R}^n$ or $\mathbb{R}^m$?]

<u>Proof</u> Let A be an $m \times n$ matrix. [Show that Col A is a subspace of $\mathbb{R}^m$.]

Then the column vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are in __________ .

By a Theorem 1 Sec. 4.1, Span $\{$___________________$\}$ is a subspace of $\mathbb{R}^m$. [Careful. Which vectors are in this proof.]

Therefore, since Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} =$ ____________, it is a subspace of $\mathbb{R}^m$. ■

<u>Ex</u>: Find a matrix A such that the given set <u>***is***</u> Col A.

$$\left\{ \begin{bmatrix} a + 2b - c \\ 2b - 3d \\ -a - c + d \\ 3a - 4c \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

Hint: Write the vector in parametric vector form.

Homework: Read the table on p. 204 and Section 4.2, p. 205: #1, 3, 6, 7, 9, 12, 15, 17,18, 25(a-d, f), [31, 34]