DEF Let V be a nonempty set on which the operation of addition and scalar[1] multiplication are defined. The set V together with these operations is called a ________________ if the following 10 axioms are satisfied for $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. $\mathbf{u} + \mathbf{v} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There exists $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. For each $\mathbf{u} \in V$, there exist $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. $c\mathbf{u} \in V$
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

The elements of V (e.g. $\mathbf{u}, \mathbf{v}, \ldots$) are called ____________ .

EX: Let P_n denote the set of all polynomials of degree n or less. Verify that P_n is a vector space.

i.e. $\mathbf{p} = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, $\mathbf{q} = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ and $\mathbf{r} = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$

Then

1. $\mathbf{p} + \mathbf{q} =$
2. $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$
3. $(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$ $\surd$ LHS = RHS $= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + (a_2 + b_2 + c_2)x^2 + \ldots + (a_n + b_n + c_n)x^n$
4. Let Then $\mathbf{p} + \mathbf{0} = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + 0 = \mathbf{p}$ $\surd$
5. Let $-\mathbf{p} = -a_0 + (-a_1)x + (-a_2)x^2 + \ldots + (-a_n)x^n$ which is in P_n. Then $\mathbf{p} + (-\mathbf{p}) = 0 = \mathbf{0}$ $\surd$
6. $c\mathbf{p} = ca_0 + ca_1x + ca_2x^2 + \ldots + ca_nx^n$ which is in P_n $\surd$
7. $c(\mathbf{p}+\mathbf{q}) = c\mathbf{p}+c\mathbf{q}$ $\surd$ since $c[(a_0+b_0)+(a_1+b_1)x+\ldots+(a_n+b_n)x^n] = c[a_0+a_1x+\ldots+a_nx^n]+c[b_0+b_1x+\ldots+b_nx^n]$
8. $(c + d)\mathbf{p} = c\mathbf{p} + d\mathbf{p}$ $\surd$ since $(c + d)[a_0 + a_1x + \ldots + a_nx^n] = c[a_0 + a_1x + \ldots + a_nx^n] + d[a_0 + a_1x + \ldots + a_nx^n]$
9. $c(d\mathbf{p}) = (cd)\mathbf{p}$ $\surd$ $\ldots$
10. $1\mathbf{u} = \mathbf{u}$ $\surd$ $\ldots$

[1] If the scalars are real numbers, it is a **real vector space**. If the scalars are complex numbers, it is a **complex vector space**.

Ex: Fill in the missing axioms for the following proof that $c\mathbf{0} = \mathbf{0}$ for every scalar c.

$$\begin{aligned}
c\mathbf{0} &= c(\mathbf{0}+\mathbf{0}) \\
&= c\mathbf{0}+c\mathbf{0} \\
c\mathbf{0}+(-c\mathbf{0}) &= (c\mathbf{0}+c\mathbf{0})+(-c\mathbf{0}) \\
c\mathbf{0}+(-c\mathbf{0}) &= c\mathbf{0}+(c\mathbf{0}+-c\mathbf{0}) \\
\mathbf{0} &= c\mathbf{0}+\mathbf{0} \\
\mathbf{0} &= c\mathbf{0}
\end{aligned}$$

Ex: Using the definition, axioms, and previous theorem(s) (and properties of real numbers) prove:
If $c\mathbf{u} = \mathbf{0}$ for some nonzero scalar c, then $\mathbf{u} = \mathbf{0}$

Ex: Show that $W = \left\{(s, 2s+1) \in \mathbb{R}^2 \;\middle|\; s \in \mathbb{R}\right\}$ is not a vector space.

<u>Def</u> A _______________ of a vector space V is a subset H of V that satisfies the following properties:

0. H is a subset of V

1. $\mathbf{0} \in H$ [Alternatively, require H to be nonempty. See footnote in book after definition (p. 193)]

2. For $\mathbf{u}$ and $\mathbf{v} \in H$, then $\mathbf{u} + \mathbf{v} \in H$

3. For $\mathbf{u} \in H$ and a scalar c, $c\mathbf{u} \in H$.

H is a subspace of V is denoted

<u>**EX**</u> Show that $H = \{\mathbf{0}\}$ is a subspace of V for any vector space V.

<u>**Question:**</u> Is V itself a subspace of V?

<u>**Question**</u>: Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? Why or why not?

<u>EX</u> Let $H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Is H a subspace of $\mathbb{R}^3$?

<u>EX</u> Let $H = \left\{ \begin{bmatrix} a \\ b \\ -a \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Is H a subspace of $\mathbb{R}^3$?

Homework: Section 4.1, p. 195: #1, 3, 5, 7, 8, 19, 21, 25, 27, 29