

THEOREM Cramer's Rule

Let A be an $n \times n$ invertible matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ for the equation $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A} = \frac{|A_i(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of matrix A with the vector $\mathbf{b}$.

$$\text{i.e. } A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{b} \quad \dots \quad \mathbf{a}_n]$$

$\uparrow$ column i

PROOF Let A be _____ and let $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b}$ in $\mathbb{R}^n$.

Claim 1: $A_i(\mathbf{b})$ can be written as the product $AI_i(\mathbf{x})$

(i.e. $A_i(\mathbf{b}) = AI_i(\mathbf{x})$ where $I_i(\mathbf{x})$ is the $n \times n$ identity matrix with the i^{th} column replaced by vector $\mathbf{x}$.)

Proof (of Claim 1):

Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ denote the _____ and $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the _____.

Then

$$\begin{aligned} AI_i(\mathbf{x}) &= A[\quad \quad \quad] \\ &= [A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \dots \quad A\mathbf{x} \quad \dots \quad A\mathbf{e}_n] \\ &= [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{b} \quad \dots \quad \mathbf{a}_n] \\ &= A_i(\mathbf{b}) \quad \blacksquare (\text{Claim 1}) \end{aligned}$$

Claim 2: $\det I_i(\mathbf{x}) = x_i$

Proof (of Claim 2): Consider the $n \times n$ matrix $I_i(\mathbf{x}) =$

$$\begin{bmatrix} 1 & 0 & \dots & x_1 & 0 & \dots & 0 \\ 0 & 1 & \dots & x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & x_i & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & x_n & 0 & \dots & 1 \end{bmatrix}$$

and compute the determinant using a cofactor expansion along row i :

$$\begin{aligned} \det I_i(\mathbf{x}) &= \text{_____} \cdot \det(I_{n-1}) \quad \text{since all the other elements in the row are zero.} \\ &= x_i \cdot 1 \cdot 1 \quad \text{since the determinant of any size identity matrix is 1.} \\ &= x_i \quad \blacksquare (\text{Claim 2}) \end{aligned}$$

over $\longrightarrow$

Back to Main Proof:

From Claim 1, we have

$$\underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

$$\Rightarrow \det A_i(\mathbf{b}) = \det AI_i(\mathbf{x}) \quad \text{by taking the determinant of both sides.}$$

$$\det A_i(\mathbf{b}) = \underline{\hspace{2cm}} \quad \text{by previous theorem}$$

$$\det A_i(\mathbf{b}) = \det A \cdot x_i \quad \text{by Claim 2}$$

Then since $\underline{\hspace{2cm}}$ we can divide to get

$$\Rightarrow x_i = \frac{\det A_i(\mathbf{b})}{\det A} \quad \blacksquare$$

Ex: Use Cramer's Rule to solve

$$\begin{array}{rcl} -2x_1 & + & 4x_2 = 7 \\ 5x_1 & + & 3x_2 = 2 \end{array}$$

Note:

Application of Cramer's Rule to Differential Equations

The **LaPlace Transform** $\mathcal{L}\{f(t)\} = \int_0^\infty e^{st} f(t) dt =$

converts a (system of) linear differential equation(s) into a (system of) linear _____ equation(s).

Ex Applying the LaPlace Transform to the system of differential equations

[Details BYSC]

$$\begin{array}{llll} \frac{dx}{dt} = 2x + y & x(0) = 1 & & (s-2)x_1 - x_2 = 1 \\ & & \text{results in the following algebraic system} & \\ \frac{dy}{dt} = 3x + 4y & y(0) = 0 & & -3x_1 + (s-4)x_2 = 0 \end{array}$$

(a). Determine the value(s) of s for which the system has a unique solution.

(b). Use Cramer's Rule to find the solution.

Note: The solution to the original differential equation is found by determining the inverse LaPlace transform of x_1 and x_2 (BYSC).
 But the answer is $x(t) = L^{-1} \left\{ \frac{s-4}{(s-1)(s-5)} \right\} = \frac{3}{4}e^t + \frac{1}{4}e^{5t}$ and $y(t) = L^{-1} \left\{ \frac{3}{(s-1)(s-5)} \right\} = -\frac{3}{4}e^t + \frac{3}{4}e^{5t}$

Recall the formula for finding the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $A^{-1} =$

Is there a similar formula for $n \times n$ matrices?

DEF Let A be an $n \times n$ matrix. The _____, denoted $\text{adj } A$ is the _____ given by

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & c_{32} & \cdots & c_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix} \quad \text{where } c_{ij} =$$

Note: To find the adjoint of A , replace each entry in A with _____, then _____ it.

i.e. $\text{adj } A =$

THEOREM If A is an $n \times n$ invertible matrix, then $A^{-1} = \frac{1}{\det A} \text{adj } A$.

PROOF [See book. It comes from applying Cramer's Rule.]

EX: Verify that the above theorem gives the formula for the inverse of a 2×2 matrix.

[On Board Example]

Homework (slightly different than assignment sheet): Section 3.3, p. 185: #2, 5, 7, 9, 13, 16,