

1. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -4 & 1 \\ 2 & 0 & 12 \end{bmatrix}$, if it exists.

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ -1 & -4 & 1 & 0 & 1 & 0 \\ 2 & 0 & 12 & 0 & 0 & 1 \end{array} \right] &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & -6 & 12 & -2 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 6 & -12 & 2 & 0 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -6 & 8 & 6 & -1 \end{array} \right] \\ &\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 3 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -4/3 & -1 & 1/6 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 6 & -1/2 \\ 0 & 1 & 0 & -7/3 & -2 & 1/6 \\ 0 & 0 & 1 & -4/3 & -1 & 1/6 \end{array} \right] \end{aligned} \quad \text{So } A^{-1} = \begin{bmatrix} 8 & 6 & -1/2 \\ -7/3 & -2 & 1/6 \\ -4/3 & -1 & 1/6 \end{bmatrix}$$

But why does this method work? [Rhetorical Question... By the end of this worksheet, you will have proved why it works.]

DEF An **elementary matrix** E is one that is obtained by performing _____ on the Identity matrix.

Ex:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \quad \Rightarrow \quad \text{Elementary Matrix } E_1 =$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 3R_1 \leftrightarrow R_1 \quad \Rightarrow \quad \text{Elementary Matrix } E_2 =$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -2R_3 + R_2 \leftrightarrow R_2 \quad \Rightarrow \quad \text{Elementary Matrix } E_3 =$$

EX: Given a general 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and the three elementary matrices defined above, compute the following products.

$$(a). \quad E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$(b). \quad E_2 A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$(c). \quad E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

[Go on to answer the next two questions (any maybe start filling in the blanks of the Facts and Proof).]

Which elementary row operation transforms A into the resulting E_1A, E_2A, E_3A , respectively?

How do these elementary row operations compare with the ones used to transform I into E_1, E_2, E_3 , respectively?

FACT The result of performing an elementary row operation on a $m \times n$ matrix A can be written as the product _____ where E is the $m \times m$ elementary matrix corresponding to performing _____ on the identity matrix I_m . [Given w/o proof – but should be clear from previous example.]

FACT Each elementary matrix is invertible. [Given w/o proof – but E^{-1} is clearly found by performing the operation necessary to take E back to the identity I .]

Theorem An $n \times n$ matrix is invertible if and only if A is row equivalent to I_n .

PROOF \Rightarrow : Let A be an _____ $n \times n$ matrix. [Show that it is row equivalent to I_n .]
i.e. Show that the unique RREF is _____.]

Then for each _____ in \mathbb{R}^n , $A\mathbf{x} = \mathbf{b}$ has a (unique) solution (by Theorem _____).

Thus, A has a _____ in every row, so there are n pivots.

Since A is $n \times n$, there are pivots in _____, as well.

Thus the n pivot positions are along the diagonal.

Therefore, the _____ form of A is I_n , i.e. $A \sim I_n$.

\Leftarrow : [Stop here. We will prove this direction together in class.]

NOTE: From the previous proof, we saw that the sequence of row operations that reduce A to I can be written in the product form _____ for the elementary matrices E_1, E_2, \dots, E_p corresponding to the row operations in order $1, 2, \dots, p$.

COROLLARY If A is an invertible $n \times n$ matrix, then any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1} .

PROOF Since A is invertible, we have by the previous theorem that $A =$ _____ for some set of elementary matrices corresponding to a sequence of elementary row operations.

Taking the inverse of both sides

$$\begin{aligned} A^{-1} &= ((E_p \cdots E_2 E_1)^{-1})^{-1} \\ \Rightarrow &= \text{_____} \text{ since the inverse of an inverse matrix returns the original matrix.} \end{aligned}$$

Since multiplying by the identity I_n does not change the matrix, multiply the RHS by I_n to obtain

$$\text{_____} = (E_p \cdots E_2 E_1) I_n.$$

In other words, the same sequence of elementary row operations applied in the order _____, which reduce A to I_n will reduce I_n to _____. ■

[NOTE: This corollary proves why $[A|I] \rightarrow [I|A^{-1}]$.