**1.** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. $T$ is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

<u>Proof</u> Let $T$ be defined as above.

$\Longrightarrow$: Let $T$ be one-to-one.                     [Show that $T(\mathbf{x}) = \mathbf{0}$ has _____ .]

By definition of one-to-one, if $\mathbf{b}$ is in $\mathbb{R}^m$, then there is at most one solution $\mathbf{x}$ in $\mathbb{R}^n$ such that _____

Specifically, since the zero vector $\mathbf{0}$ is in _____ and $T$ is one-to-one,

there is _____ solution $\mathbf{x}$ in $\mathbb{R}^n$ such that _____ .                [By the definition of one-to-one]

Since $T$ is linear, _____ is always a solution to $T(\mathbf{x}) = \mathbf{0}$. i.e. $T(\mathbf{0}) = \mathbf{0}$.

Since there is _____ solution, $\mathbf{x} = \mathbf{0}$ is the *only* solution.

$\Longleftarrow$: Let $T(\mathbf{x}) = \mathbf{0}$ have only the trivial solution.                [Show that $T$ is one-to-one.]

BWOC, suppose _____ .

Then there exists a vector $\mathbf{b}$ in $\mathbb{R}^m$ and two _____ vectors $\mathbf{u}$ and $\mathbf{v}$ such that $T(\mathbf{u}) = \mathbf{b}$ and $T(\mathbf{v}) = \mathbf{b}$.

Then
$$\begin{aligned} T(\mathbf{u} - \mathbf{v}) \;&=\; \text{_____} \qquad \text{since } T \text{ is linear.} \\ &=\; \mathbf{b} - \mathbf{b} \\ &=\; \mathbf{0} \end{aligned}$$

i.e. $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, which has only the _____ solution [by the given statement (see $\Longleftarrow$:)].

So $\mathbf{u} - \mathbf{v} = \mathbf{0}$ is this trivial solution.

$\Rightarrow$ _____ $\rightarrow\!\!\times\!\!\leftarrow$

Therefore, $T$ must be one-to-one. $\blacksquare$

**2.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined as $T(\mathbf{x}) = A\mathbf{x}$. Then

**(a).** $T$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of $A$ span $\mathbb{R}^m$.

**(b).** $T$ is one-to-one iff the columns of $A$ are linearly independent.

<u>PROOF</u> Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined as $T(\mathbf{x}) = A\mathbf{x}$

**(a).** By a previous theorem (sec 1.4),

the columns of $A$ span $\mathbb{R}^m$ iff for _____ the equation $A\mathbf{x} = \mathbf{b}$ has a _____ (i.e. at least one solution).

But since $A\mathbf{x} = \mathbf{b}$ is equivalent to the equation _____ , the statement becomes:

The columns of $A$ span $\mathbb{R}^m$ iff for each $\mathbf{b}$ in $\mathbb{R}^m$ the equation _____ has at least one solution.

Therefore, by definition of _____ , $T$ is onto $\mathbb{R}^m$ iff the columns of $A$ span $\mathbb{R}^m$. ∎

**(b).** From sec. 1.7, the columns of $A$ are linearly independent iff $A\mathbf{x} = \mathbf{0}$ has _____ .

$\Rightarrow$ The columns of $A$ are linearly independent iff _____ has only the trivial solution.

From the previous theorem, $T$ is _____ iff $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Therefore, combining the last 2 statements:

$T$ is one-to-one iff the columns of $A$ are _____ . ∎

[Now we can complete the Invertible Matrix Theorem.]