1. A set of p vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

<u>PROOF</u>
Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be in $\mathbb{R}^n$ where $p > n$. Then let $A = [\ \mathbf{v}_1\ \ \mathbf{v}_2\ \ \cdots\ \ \mathbf{v}_p\ ]$, which is a matrix of size ____________ .

Since A has ________ rows, there can be at most ________ pivots, which is less than the number of columns.

Then there must be a __________________ corresponding to each of the columns without a pivot.

Thus, $A\mathbf{x} = \mathbf{0}$ has a ________________________ .

Therefore the columns of A are linearly dependent (by (3) on p. 57). ■

[Make sure you understand how this equation (3) relates to the definitions of linear dependence and independence.]

2. If a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

<u>PROOF</u>
Let S be defined as above.

WLOG, suppose $\mathbf{v}_1 =$ ________

Let $c_1 \neq 0$ and $c_2, c_3, \ldots, c_p = 0$.

Then

$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p &= c_1\mathbf{v}_1 + \underline{\qquad}\,\mathbf{v}_2 + \underline{\qquad}\,\mathbf{v}_3 + \ldots + \underline{\qquad}\,\mathbf{v}_p \\ &= c_1\,\mathbf{0} + \mathbf{0} + \mathbf{0} + \ldots + \mathbf{0} \qquad \text{since } \mathbf{v}_1 = \underline{\qquad} \text{ and scalar multiplication.} \\ &= \mathbf{0} \end{aligned}$$

Therefore, the set is linearly dependent since there exists a ____________________ , namely c_1, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$. ■

3. A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors in linearly dependent iff at least one of the vectors in S is a linear combination of the other vectors.
<u>PROOF</u>
Let S be defined as above.

$\Longrightarrow$: Done in class

$\Longleftarrow$: Let one vector be a _______________________ of the other vectors.

i.e. $\exists$____________ such that $\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_{k-1}\mathbf{v}_{k-1} + c_{k+1}\mathbf{v}_{k+1} + \ldots + c_p\mathbf{v}_p$

By subtraction $\mathbf{0} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_{k-1}\mathbf{v}_{k-1}$___________ $+ c_{k+1}\mathbf{v}_{k+1} + \ldots + c_p\mathbf{v}_p$

So the weight $c_k =$ _________.

Which means at least one weight is ______________ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0}$.

Therefore, by definition, S is _______________________ . ■

<u>COROLLARY</u> If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$ then there exists $\mathbf{v}_j$ in S with $j > 1$ such that $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

<u>PROOF</u> [Feel free to start, but Dr. Crawford will do this one.]