Interchange, Scaling, and Replacement are called elementary row operations for matrices.

<u>Def</u> Two matrices are row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other.

<u>Def</u> A leading entry of a row is the left-most nonzero entry in that row.

<u>Ex</u>: (from previous worksheet) $\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -4 & 5 & 33 \\ 2 & -1 & 1 & 13 \end{bmatrix}$

We found a row equivalent matrix of the one above that was in a "good" form (step 4 of previous worksheet):

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 9 & 45 \end{bmatrix}$$

But we went further to find a row equivalent matrix in an even "better" form (step 7):

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

<u>Ex</u>: $\begin{bmatrix} -8 & -4 & -6 & -2 & 4 \\ 0 & 0 & 3 & 6 & 3 \\ 4 & 2 & 1 & 0 & -4 \\ 0 & 0 & 2 & 1 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 4 & 2 & 3 & 1 & -2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

<u>Def</u> A rectangular matrix is in (row) echelon form (REF) if it has the following 3 properties: [Guess]

1. Any row of all zeros are below the nonzero rows.

2. The position of the leading entry in each row is to the right of the leading entry of the row above it.

3. All entries in a column <u>below</u> a leading entry are zero.

<u>Def</u> Furthermore, it is in (row) reduced echelon form (RREF) if these 2 additional properties hold:

4. The leading entry in each nonzero row is 1.

5. Each leading entry (1) is the <u>only</u> nonzero entry in its column. (i.e. 0's above <u>and</u> below the leading entry)

<u>Theorem</u> The Reduced (Row) Echelon Form is __unique__ for any given matrix.

<u>Def</u> A __pivot position__ in a matrix is the __location__ of a leading 1 in reduced echelon form.

<u>Def</u> A __pivot column__ is a column that contains a __pivot position__ .

<u>Def</u> A __pivot__ is a nonzero number in the pivot position used to create zeros in the rows above and below.

Row Reduction Algorithm
(variant of Gaussian Elimination)

<u>Forward Phase</u>
(to echelon form)

Step 1
Locate the __leftmost__ nonzero column and note:

- This is a __pivot column__
- The __pivot position__ is at the top of this column

Ex:

$$\begin{array}{rcrcrcrcr} & & -5x_2 & + & x_3 & + & x_4 & = & 5 \\ 2x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & = & 11 \\ x_1 & - & 2x_2 & + & 2x_3 & + & 2x_4 & = & 8 \\ 5x_1 & & & & & + & 2x_4 & = & 20 \end{array}$$

$$\begin{bmatrix} 0 & -5 & 1 & 1 & 5 \\ 2 & 1 & 3 & 3 & 11 \\ 1 & -2 & 2 & 2 & 8 \\ 5 & 0 & 0 & 2 & 20 \end{bmatrix}$$

Step 2
Choose a nonzero number in this column to be the __pivot__ .

- Choose wisely
- If necessary, interchange rows to move it to the pivot position
- (optional) Scale row to get a 1 in the pivot position.

Step 3

Use ___Row Operations___ to get all zero entries below the pivot

Step 4

Ignore/Cover all rows above and including the pivot position.

___Repeat steps 1-4 on the submatrix___ until echelon form attained.

BACKWARD PHASE

(to ***reduced*** echelon form)

Step 5

Locate the rightmost ___pivot___ .

(a). Scale row to make ___pivot = 1___ .

(b). Use Row Operations to get ___zero entries___ above the pivot.

(c). Locate the next rightmost pivot. Repeat steps 5(a)-5(b) until reduced echelon from is attained.

$$\begin{bmatrix} 1 & -2 & 2 & 2 & 8 \\ 0 & 5 & -1 & -1 & -5 \\ 0 & 0 & -8 & -6 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[Extra space for previous problem, if needed.]

Linear System ⇒ Matrix ⇒ REF ⇒

$$\begin{array}{rcrcrcrcr} & & -5x_2 & + & x_3 & + & x_4 & = & 5 \\ 2x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & = & 11 \\ x_1 & - & 2x_2 & + & 2x_3 & + & 2x_4 & = & 8 \\ 5x_1 & & & & & + & 2x_4 & = & 20 \end{array} \Rightarrow \begin{bmatrix} 0 & -5 & 1 & 1 & 5 \\ 2 & 1 & 3 & 3 & 11 \\ 1 & -2 & 2 & 2 & 8 \\ 5 & 0 & 0 & 2 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 2 & 2 & 8 \\ 0 & 5 & -1 & -1 & -5 \\ 0 & 0 & -8 & -6 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

RREF ⇒ System ⇒ Solution

$$\begin{bmatrix} 1 & 0 & 0 & 2/5 & 4 \\ 0 & 1 & 0 & -1/20 & -3/4 \\ 0 & 0 & 1 & 3/4 & 5/4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{rrrcrcr} x_1 & & & + & 2/5x_4 & = & 4 \\ & x_2 & & - & 1/20x_4 & = & -3/4 \\ & & x_3 & + & 3/4x_4 & = & 5/4 \\ & & & & 0 & = & 0 \end{array} \Rightarrow \left\{ \begin{array}{rcrcr} x_1 & = & 4 & - & 2/5x_4 \\ x_2 & = & -3/4 & + & 1/20x_4 \\ x_3 & = & 5/4 & - & 3/4x_4 \\ x_4 & is & free & & \end{array} \right.$$

<u>Def</u> The basic (or leading) variables are the variables corresponding to the pivot columns.

<u>Ex</u>: x_1, x_2, x_3 are basic variables

<u>Def</u> Any remaining variables not associated with the pivot columns are called free variables.

<u>Ex</u>: x_4 is a free variable

$\Longrightarrow$ Solution: