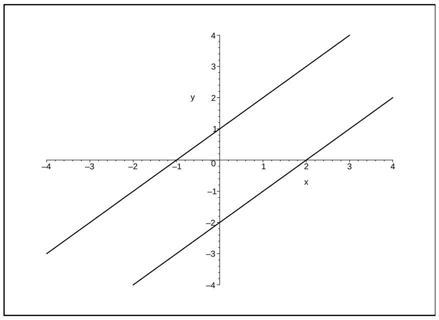
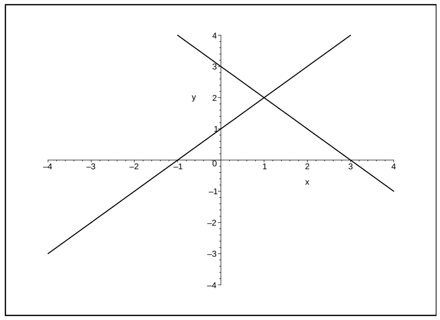
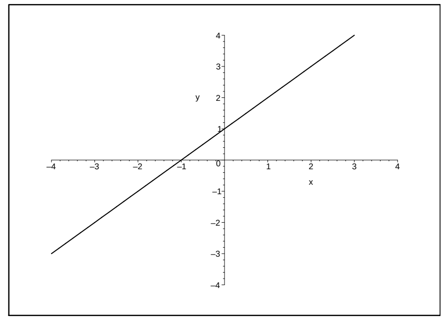
Graph 2 lines in $\mathbb{R}^2$ and only 3 things can happen:







lines are parallel

lines intersect at exactly one point

lines are the same

Given **equations** for 2 lines,
these graphical possibilities show the 3 possibilities for a solution to both equations

$$\begin{aligned} y &= x + 1 \\ y &= x - 2 \end{aligned}$$

$$\begin{aligned} y &= x + 1 \\ y &= -x + 3 \end{aligned}$$

$$\begin{aligned} x - y &= -1 \\ -2x + 2y &= 2 \end{aligned}$$

no solution

no point
makes both eqns true

unique solution

(1,2) is only point that
makes both eqns true

infinitely many solutions

all points on the line
make both eqns true

inconsistent

consistent

consistent and dependent

Def A <u>linear equation</u> in n variables $x_1, x_2, x_3, ..., x_n$ is of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots a_nx_n = b$$

where $a_1, a_2, a_3, ..., a_n$ and b are <u>real or complex numbers</u> and $a_1, a_2, a_3, ..., a_n$ are the <u>coefficients</u>.

Ex: $y = x + 1$ Ex: $2\sqrt{3}x_1 - x_2 + (10 - 3i)x_4 = 7$ Ex: $x_1 = 3(x_2 + 4x_3)$

Def A **system of linear equations** is a collection of one or more linear equations involving the same variables.

Ex: (3 examples on page 1) Ex: $\begin{array}{rcrcrcr} 3x_1 & & & - & x_3 & = & 10 \\ x_1 & + & x_2 & + & 4x_3 & = & 6 \end{array}$ Ex: $\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ 2x_1 & - & x_2 & + & x_3 & = & 13 \\ & & -4x_2 & + & 5x_3 & = & 33 \end{array}$

Def A __solution of a system__ is a list of numbers $(s_1, s_2, s_3, ..., s_n)$ that make all of the equations of a system __true__ when substituted for $x_1, x_2, x_3, ..., x_n$, respectively.

Def The __solution set__ is the set of all possible solutions to a system.

Def Two linear systems are __equivalent__ if they have the same solution set.

Ex: $(3, 7, -1)$ is a solution to the system $\begin{array}{rcrcrcr} 3x_1 & & & - & x_3 & = & 10 \\ x_1 & + & x_2 & + & 4x_3 & = & 6 \end{array}$

Ex: $\{(k, k+1) | k \in \mathbb{R}\}$ is the solution set to the system $\begin{array}{rcrcr} x & - & y & = & -1 \\ -2x & + & 2y & = & 2 \end{array}$

Def A system of linear equations is called

- __inconsistent__ if it has no solutions.
- __consistent__ if it has either one or infinitely many solutions.

  Furthermore, if it has infinitely many solutions the system is called __dependent__ .

Important Note:

<u>Def</u> A **<u>matrix</u>** is a rectangular array of elements (often numbers)

An $m \times n$ matrix has m ___rows___ and n ___columns___

A system of linear equations can be represented by two types of matrices:

<u>Ex</u>: <u>Linear System</u>

$$\begin{aligned} 3x_1 \phantom{{}+2x_2} &+ 3x_3 = -6 \\ x_1 + 2x_2 &- x_3 = 0 \\ 2x_1 - x_2 &+ 5x_3 = 1 \end{aligned}$$

1. <u>Coefficient Matrix</u>

coefficients of each variable form the columns

$$\begin{bmatrix} 3 & 0 & 3 \\ 1 & 2 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

2. <u>Augmented Matrix</u>

the coefficient matrix with an additional column containing the RHS constants

$$\begin{bmatrix} 3 & 0 & 3 & -6 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 5 & 1 \end{bmatrix}$$

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3 Operations on systems/matrices that yield an equivalent system/matrix $\Longrightarrow$ Same solution set

(a). **<u>Interchange</u>**: Interchange 2 equations/rows

(b). **<u>Scaling</u>**: Multiply the entire equation/row by a nonzero constant

(c). **<u>Replacement</u>**: Replace one equation/row by the sum of itself and a multiple of another equation/row

Ex:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 & \text{①} \\ -4x_2 + 5x_3 &= 33 & \text{②} \\ 2x_1 - x_2 + x_3 &= 13 & \text{③} \end{aligned}$$

Ex:

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -4 & 5 & 33 \\ 2 & -1 & 1 & 13 \end{bmatrix}$$

1. Interchange equations ② and ③

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 & \text{①} \\ 2x_1 - x_2 + x_3 &= 13 & \text{②} \\ -4x_2 + 5x_3 &= 33 & \text{③} \end{aligned}$$

1. Interchange rows 2 and 3 $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 2 & -1 & 1 & 13 \\ 0 & -4 & 5 & 33 \end{bmatrix}$$

2. Replace equation ② with: $-2 \cdot$ ① $+$ ②

$$\begin{aligned} -2x_1 - 4x_2 - 6x_3 &= -28 \\ 2x_1 - x_2 + x_3 &= 13 \\ \hline -5x_2 - 5x_3 &= -15 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 & \text{①} \\ -5x_2 - 5x_3 &= -15 & \text{②} \\ -4x_2 + 5x_3 &= 33 & \text{③} \end{aligned}$$

2. Replace row 2 with: $-2 \cdot \text{row1} + \text{row2}$ $-2R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & -5 & -5 & -15 \\ 0 & -4 & 5 & 33 \end{bmatrix}$$

3. Scale equation ② by multiplying by $-\frac{1}{5}$

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & + & 3x_3 & = & 14 & \quad ① \\ & & x_2 & + & x_3 & = & 3 & \quad ② \\ & & -4x_2 & + & 5x_3 & = & 33 & \quad ③ \end{array}$$

3. Scale row 2 by multiplying by $-\frac{1}{5}$ $\quad -\frac{1}{5}R_2 \to R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 1 & 3 \\ 0 & -4 & 5 & 33 \end{bmatrix}$$

4. Replace equation ③ with: $4 \cdot$ ② + ③

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & + & 3x_3 & = & 14 & \quad ① \\ & & x_2 & + & x_3 & = & 3 & \quad ② \\ & & & & 9x_3 & = & 45 & \quad ③ \end{array}$$

We could stop here and solve by back substitution.

4. Replace row 3 with: $4 \cdot \text{row2} + \text{row3}$ $\quad 4R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 9 & 45 \end{bmatrix}$$

But we want an easier form.

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5. Scale equation ③ by multiplying by $\frac{1}{9}$

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & + & 3x_3 & = & 14 & \quad ① \\ & & x_2 & + & x_3 & = & 3 & \quad ② \\ & & & & x_3 & = & 5 & \quad ③ \end{array}$$

5. Scale row 3 by multiplying by $\frac{1}{9}$ $\quad \frac{1}{9}R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 14 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

6. Replace equation ② with: $-1 \cdot$ ③ + ②

Replace equation ① with: $-3 \cdot$ ③ + ①

$$\begin{array}{rcrcrcrl} x_1 & + & 2x_2 & & & = & -1 & \quad ① \\ & & x_2 & & & = & -2 & \quad ② \\ & & & & x_3 & = & 5 & \quad ③ \end{array}$$

6. Replace row 2 with: $-1 \cdot \text{row3} + \text{row2}$ $\quad -R_3 + R_2 \to R_2$

Replace row 1 with: $-3 \cdot \text{row3} + \text{row1}$ $\quad -3R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

7. Replace equation ① with: $-2 \cdot$ ② + ①

$$\begin{array}{rcrcrcrl} x_1 & & & & & = & 3 & \quad ① \\ & & x_2 & & & = & -2 & \quad ② \\ & & & & x_3 & = & 5 & \quad ③ \end{array}$$

7. Replace row 1 with: $-2 \cdot \text{row2} + \text{row1}$ $\quad -2R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$