Name: Key

Math 362 Linear Algebra - Crawford

Exam 2

15 November 2017

Books and notes (in any form) are not are allowed. You may use a calculator – but please indicate when you use the matrix functions on the calculator. Put all of your work and answers on the separate paper provided and staple this cover sheet on top. Show all your work for credit. ***Good luck!*** Calculator # ________

1. (16 pts) Given the system $\begin{array}{rcrcl} 3sx_1 & - & 5x_2 & = & 3 \\ 9x_1 & + & 5sx_2 & = & 2 \end{array}$, which contains the parameter s,

(a). Use Cramer's Rule to find the solution.

(b). Determine the value(s) of s for which the system has a unique solution.

2. (16 pts) Given $A = \begin{bmatrix} 4 & 4 & -2 & 6 & 0 \\ 1 & 1 & 2 & 4 & 0 \end{bmatrix}$.

(a). Find a basis for Col A. **(b).** Find a basis for Nul A.

3. (16 pts) Let H be the set of all matrices in $M_{n\times n}$ such that $A^2 = A$. That is, $H = \{A \text{ in } M_{n\times n} \mid A^2 = A\}$. Determine whether H is a subspace of $M_{n\times n}$. If it is not a subspace, clearly show **all** *subspace* properties that do not hold.

4. (8 pts) Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 9 \\ 3 \end{bmatrix}$. Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

It can be verified that $2\mathbf{v}_1 + 5\mathbf{v}_2 - 2\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for H. [Justify your answer.]

5. (9 pts) Determine whether the following statements are true or false. If the statement is false, correct the statement and/or clearly explain why it is false. [You may answer these questions on this cover sheet, if you would like.]

(a). For an $n \times n$ matrix A, $\det(kA) = k^n \det(A)$.

$\boxed{\text{True}}$

(b). For an $n \times n$ matrix A, if the $\det(A) = 0$, then two rows or two columns are the same, or a row or a column is zero.

$\boxed{\text{False}}$ The columns can be linearly dependent in other ways.

(c). For $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$, the determinant is given by $|A| = a\begin{vmatrix} f & g & h \\ j & k & l \end{vmatrix} - e\begin{vmatrix} b & c & d \\ j & k & l \end{vmatrix} + i\begin{vmatrix} b & c & d \\ f & g & h \end{vmatrix}$.

$\boxed{\text{False}}$ Determinants are defined only for $\underline{n\times n}$ matrices

6. (8 pts) Answer the following questions and ***briefly*** explain your answer. [You may answer these questions on this cover sheet, if you would like.]

(a). If A is a 5×5 matrix and the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^5$, is it possible that for some $\mathbf{b}$, the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution? Why or why not?

Square

$\boxed{\text{No}}$

$A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^5$.

$\Rightarrow$ Columns of A span $\mathbb{R}^5$

$\Rightarrow$ All statements of the IMT are true (A is invertible)

$\Rightarrow$ $A\vec{x} = \vec{b}$ has a unique sol$^{\underline{n}}$.

(b). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the property that $T(\mathbf{u}) = T(\mathbf{v})$ for some distinct pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Can T map $\mathbb{R}^n$ onto $\mathbb{R}^n$? Why or why not?

$\boxed{\text{No}}$

Since $T(\vec{u}) = T(\vec{v})$ for distinct $u, v \in \mathbb{R}^n$,

then T is not one-to-one.

Since $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, it has the matrix representation $T(\vec{x}) = A\vec{x}$, for $n \times n$ A.

$\Rightarrow$ A is not invertible

$\Rightarrow$ T is not onto $\mathbb{R}^n$.

7. (30 pts) Prove 3 of the following. Clearly state theorems and properties that you use.

BONUS: You may do (or attempt) all four options and each will be graded out of 10 points. Whichever three you score higher on will be your base grade. Any points from the fourth problem will be cut in third and added to your base grade.

(a). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Prove that the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is also linearly dependent.

(b). (New-ish) If the columns of an $n \times n$ matrix A are linearly independent, prove that the columns of A^2 span $\mathbb{R}^n$.

(c). (New) Let B be an $m \times m$ invertible matrix and let A be an $m \times n$ matrix. If $\mathbf{x}$ is in Nul (BA), prove that $\mathbf{x}$ is in Nul (A).

(d). (New) Let A be an invertible $n \times n$ matrix. Prove that $\det(\text{adj } A) = (\det A)^{n-1}$. [Do ***not*** try to write out the adjoint for an arbitrary $n \times n$ matrix.]

Exam 2 Key

p1/

1. $\begin{bmatrix} 3s & -5 \\ 9 & 5s \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(a) $x_1 = \dfrac{\begin{vmatrix} 3 & -5 \\ 2 & 5s \end{vmatrix}}{\begin{vmatrix} 3s & -5 \\ 9 & 5s \end{vmatrix}} = \dfrac{15s+10}{15s^2+45} = \dfrac{5(3s+2)}{{}_3 15(s^2+3)} = \boxed{\dfrac{3s+2}{3(s^2+3)}}$ x_1

$x_2 = \dfrac{\begin{vmatrix} 3s & 3 \\ 9 & 2 \end{vmatrix}}{15(s^2+3)} = \dfrac{6s-27}{15(s^2+3)} = \dfrac{3(2s-9)}{{}_5 15(s^2+3)} = \boxed{\dfrac{2s-9}{5(s^2+3)}}$ x_2

(b) $\boxed{\text{Unique sol}^{n} \text{ for all } s}$

since $\det A = 15(s^2+3) \neq 0$ for all s

2. $A = \begin{bmatrix} 4 & 4 & -2 & 6 & 0 \\ 1 & 1 & 2 & 4 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

pivot col^{n}s.

x_2, x_4, x_5 free var's

(a) $\boxed{\left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}}$ (Columns 1 & 3 of orig. matrix)

Basis for Col A.

(b) $x_1 + x_2 + 2x_4 = 0$
$x_3 + x_4 = 0$
x_2, x_4, x_5 free

$\Rightarrow$

$x_1 = -x_2 - 2x_4$
$x_2 = x_2$
$x_3 = -x_4$
$x_4 = x_4$
$x_5 = x_5$

$\Rightarrow$ Soln^{s} to $A\vec{x} = \vec{b}$

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul A is $\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$

p2/

3. $H = \{A \text{ in } M_{n\times n} \mid A^2 = A\}$ Is $H \subseteq M_{n\times n}$?

0. H is clearly a subset of $M_{n\times n}$. ✓

1. Is $\vec{0} \in H$? Let O_n be the $n\times n$ O matrix, then $O_n^2 = O_n$ ie $O_n \in H$ (ie $O_n = \vec{0} \in H$)

2. Let $A, B \in H$ [Is $A+B$ in H?]

$\Rightarrow A^2 = A$ and $B^2 = B$ [Does $(A+B)^2 = A+B$?]

$(A+B)^2 = (A+B)(A+B)$

$= A^2 + AB + BA + B^2$

$= A + AB + BA + B \neq A+B \Rightarrow$ $\boxed{\text{Property 2 does not hold}}$

$A+B \notin H$

3. Let $A \in H$ and c be a scalar. [Is $cA \in H$?]

$\Rightarrow A^2 = A$ [Is $(cA)^2 = cA$?]

$(cA)^2 = c^2A^2 = c^2A \neq cA \;\therefore\; (cA) \notin H.$ $\boxed{\text{Property 3 does not hold}}$

$\therefore$ $\boxed{H \text{ is not a subspace of } M_{n\times n}.}$

4. $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\underbrace{2\vec{v}_1 + 5\vec{v}_2 - 2\vec{v}_3 = \vec{0}}_{\text{Linearly Dependent}}$

$2\vec{v}_3 = 2\vec{v}_1 + 5\vec{v}_2$

$\vec{v}_3 = \vec{v}_1 + \frac{5}{2}\vec{v}_2$ ie $\vec{v}_3$ is a linear comb. of $\vec{v}_1$ & $\vec{v}_2$

$\Rightarrow \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = H$

$\boxed{\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}}$ is a basis for H since

① it spans H

② it is linearly independent since it contains only 2 vectors which are not scalar multiples of each other.

Note: $\mathcal{B}_2 = \{\vec{v}_1, \vec{v}_3\}$ or $\mathcal{B}_3 = \{\vec{v}_2, \vec{v}_3\}$ are also acceptable answers.

7(a) Proof

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dep. set in $\mathbb{R}^n$.

[Show $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is also lin dep.]

then $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ for c_1, c_2, c_3 not all 0

$\Rightarrow T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = T(\vec{0})$

$\Rightarrow \underbrace{c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + c_3T(\vec{v}_3) = \vec{0}}$ by properties of linear transformations

Since this eqn holds for the same c_1, c_2, c_3 not all 0, then $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent (by definition). ∎

(b) Proof

Let the columns of an $\underline{n \times n}$ matrix A be linearly independent. [Show that col^{n}s of A^2 span $\mathbb{R}^n$]

$\Rightarrow$ A is invertible by the IMT.

$\Rightarrow$ $\det A \neq 0$ by IMT

$\Rightarrow \det(A^2) = \underbrace{(\det A)(\det A)}_{\text{by properties of determinants}} \neq 0$

$\Rightarrow$ A^2 is invertible by IMT. (Note: A^2 is also $n \times n$)

$\therefore$ The col^{n}s of A span $\mathbb{R}^n$ by IMT. ∎

Proof

p4/

7 (c) Let B be an $m \times m$ invertible matrix.

Let A be an $m \times n$ matrix.

Let $\vec{x} \in Nul(BA)$. [Show $\vec{x} \in Nul\, A$]

Then $(BA)\vec{x} = \vec{0}$ since $\vec{x}$ in $Nul(BA)$

$\Rightarrow B(A\vec{x}) = \vec{0}$

$\Rightarrow B^{-1}B(A\vec{x}) = B^{-1}\vec{0}$ since B^{-1} exists (given)

$\Rightarrow A\vec{x} = \vec{0}$ by properties of matrix mult.

$\Rightarrow \vec{x} \in Nul(A)$ ∎

Proof

(d) Let A be an invertible $n \times n$ matrix. [Show $\det(\text{adj} A) = (\det A)^{n-1}$]

Consider property $A^{-1} = \underbrace{\frac{1}{\det A}}_{\text{scalar}} \underbrace{\text{adj}\, A}_{\text{matrix}}$

$\Rightarrow (\det A)\, A^{-1} = \text{adj}\, A$

$\Rightarrow \det\big(\underbrace{(\det A)}_{\text{scalar}} A^{-1}\big) = \det(\text{adj}\, A)$

$\Rightarrow \underbrace{(\det A)^n \det(A^{-1})}_{\text{By prop's of determinants (See T/F \#5a)}} = \det(\text{adj}\, A)$

$\Rightarrow (\det A)^n \cdot \frac{1}{\det A} = \det(\text{adj}\, A)$

$(\det A)^{n-1} = \det(\text{adj}\, A)$ ∎