Math 362 - Linear Algebra - Exam 1 Key p. 1/

1. $A = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -3 \\ 1 & 2 \\ 2 & 0 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(a) $\underset{(2\times2)\ (2\times3)}{AB^T} = \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ -3 & 2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & -1 & 2 \\ -16 & 6 & -4 \end{bmatrix}}$

(b) $\underbrace{\vec{x}\vec{x}^T}_{(3\times1)(1\times3)} + \underbrace{\vec{x}^T\vec{x}}_{(1\times3)(3\times1)}$

$\Rightarrow 3\times3$ $\quad\Rightarrow 1\times1$

Not possible to add, since $\vec{x}\vec{x}^T$ is a 3×3 matrix and $\vec{x}^T\vec{x}$ is a 1×1 matrix.

2. $x_1 + +2x_2 + x_4 = 4$
$x_3 - 3x_4 = 2$
$x_1 + 2x_2 + 2x_3 - 5x_4 = 8$
$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 0 & 1 & -3 & 2 \\ 1 & 2 & 2 & -5 & 8 \end{array}\right]$

$\xrightarrow{\text{RREF}}$ Calculator $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 4 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ ← consistent.

x_2 free $\quad x_4$ free

$x_1 + 2x_2 + x_4 = 4$
$x_3 - 3x_4 = 2$
x_2, x_4 free

$$\left.\begin{aligned} x_1 &= -2x_2 - x_4 + 4 \\ x_2 &= x_2 \\ x_3 &= 3x_4 + 2 \\ x_4 &= x_4 \end{aligned}\right.$$

$$\Rightarrow \boxed{\vec{x} = \underbrace{x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}}_{\text{sol}^{\underline{n}} \text{ to homog. system}} + \underbrace{\begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}}_{\text{particular sol}^{\underline{n}}}}$$

3. $A + B$ are invertible.

(a) Solve for X: $AXB = (AB)^2$

$$AXB = (AB)(AB)$$
$$A^{-1}AXB = A^{-1}ABAB$$
$$IXB = IBAB$$
$$XB = BAB$$
$$XBB^{-1} = BABB^{-1}$$
$$XI = BAI$$
$$X = BA$$

OR

$$AXB = (AB)(AB)$$
$$AXB = A(BA)B$$
$$X = BA$$

Lots of possibilities

(b) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{(1)(1)-2(0)}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \Rightarrow B^{-1} = \frac{1}{(2)(1)-3(0)} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -3/2 \\ 0 & 1 \end{bmatrix}$

$A+B = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \qquad \Rightarrow A^{-1}+B^{-1} = \begin{bmatrix} 3/2 & -3/2 \\ -2 & 2 \end{bmatrix}$

$(A+B)^{-1}$ DNE since $\det(A+B) = (3)(2)-(3)(2) = 0$

So, clearly $(A+B)^{-1}$ can't equal $A^{-1}+B^{-1}$

Another example:

or know $I^{-1} = I$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{(1)(1)-(0)(0)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B^{-1} = \frac{1}{(1)(4)-(2)(3)}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

$\Rightarrow A^{-1}+B^{-1} = \begin{bmatrix} -1 & 1 \\ 3/2 & 1/2 \end{bmatrix}$

$A+B = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix} \Rightarrow (A+B)^{-1} = \frac{1}{(2)(5)-(2)(3)}\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 5/4 & -1/2 \\ -3/4 & 1/2 \end{bmatrix}$

Not the same.

Exam1 p3

4. $$\left[\begin{array}{ccc|ccc} 1 & x & x & 1 & 0 & 0 \\ 0 & 1 & x & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-xR_3+R_2 \to R_2]{-xR_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & x & 0 & 1 & 0 & -x \\ 0 & 1 & 0 & 0 & 1 & -x \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-xR_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & x^2-x \\ 0 & 1 & 0 & 0 & 1 & -x \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

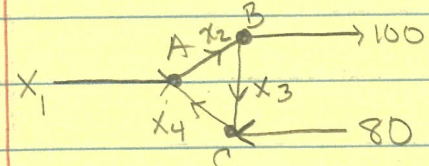
So $$\boxed{A^{-1} = \begin{bmatrix} 1 & -x & x^2-x \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}}$$

5. $$\left[\begin{array}{ccc|c} 2 & 5 & -1 & 4 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & 4 \end{array}\right] \xrightarrow[\text{Calculator}]{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$ ← Not Consistent

(a) There is no sol^n to the associated eqn $A\vec{x} = \vec{u}$

$\therefore$ $\vec{u}$ is not in the set spanned by the columns of A.

(b) If you consider the associated homogeneous eqn $A\vec{x}=0$, the sol^n contains a free variable x_3. Therefore, the columns of A are **linearly dependent**. Note: There are other possible justifications/explanations

6. 

Node	Inflow = Outflow
A	$x_1 + x_4 = x_2$
B	$x_2 = x_3 + 100$
C	$x_3 + 80 = x_4$
overall system	$x_1 + 80 = 100$

ie
$$\boxed{\begin{aligned} x_1 - x_2 \quad + x_4 &= 0 \\ x_2 - x_3 \quad &= 100 \\ x_3 - x_4 &= -80 \\ x_1 \quad &= 20 \end{aligned}}$$

7. (a) Proof: Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be in $\mathbb{R}^n$ and suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. [Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dep.]

Then there exists weights not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

Using the same not all zero weights c_1, c_2, c_3 the following equation holds for $c_4 = 0$

$$\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3}_{\vec{0} \text{ from above}} + \underbrace{0\vec{v}_4}_{\vec{0}} = \vec{0}$$

$\therefore \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent.

(b) Proof: Let A & B be invertible and let $AB = BA$.

Then A^{-1} & B^{-1} exist. [Show $(AB)^{-1} = A^{-1}B^{-1}$]

Consider $(AB)^{-1} = (BA)^{-1}$ since $AB = BA$

$= A^{-1}B^{-1}$ by property of inverses

[Note: Should also state that we know $(AB)^{-1}$ exists since properties of inverses tell us the product of 2 invertible matrices is also invertible].

(c) Proof: Let $\vec{w}$ be a sol^n to $A\vec{x} = \vec{b}$. ie $A\vec{w} = \vec{b}$ is true. (*)

Let $\vec{u}_h$ be a sol^n to $A\vec{x} = \vec{0}$, ie $A\vec{u}_h = \vec{0}$ is true. (**)

Let $\vec{v} = \vec{w} + \vec{u}_h$ [Show that $\vec{v}$ is a sol^n to $A\vec{x} = \vec{b}$ ie Show $A\vec{v} = \vec{b}$ is true]

Consider $A\vec{v} = A(\vec{w} + \vec{u}_h)$

$= A\vec{w} + A\vec{u}_h$ by Left Distributive Law

$= \vec{b} + \vec{0}$ by (*) and (**)

$= \vec{b}$

ie $A\vec{v} = \vec{b}$ is true. $\therefore \vec{v}$ is a sol^n to $A\vec{x} = \vec{b}$

Exam I p5/5

8. (a) $(A+B)^2 = A^2+2AB+B^2$ **FALSE**

$$(A+B)^2 = (A+B)(A+B) = A(A+B)+B(A+B)$$
$$= A^2 + \underbrace{AB+BA}_{\text{cannot combine since } AB \neq BA} + B^2$$

(b) **FALSE** A must have a pivot in every row for $A\vec{x}=\vec{b}$ to be consistent for each $\vec{b}$ in $\mathbb{R}^m$.

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{array}\right]$$ Counterexample

(c) **TRUE** You need at least 3 vectors in $\mathbb{R}^3$ to span $\mathbb{R}^3$.

(d) **TRUE** $AB = [A\vec{b}_1 \; A\vec{b}_2 \; A\vec{b}_3]$ by Def$^{\underline{n}}$

(Explanations are not required for True statements)