Numerical Methods – Using Excel to Solve by Iteration

Using finite differences to approximate a solution to a differential equation leads to a system of n+1 equations with n+1 unknowns. Such systems can be solved using a variety of techniques: *Elimination and Back Substitution*, *Matrix Inversion*, & ***Iteration***.

We will use Excel to find approximate solutions using the iteration method. The idea behind iteration is to give an initial guess for the unknown values and apply the relevant equations. Use the results as the new guess and apply the equations again. Continue this process until the solution converges (i.e. the values do not change anymore)[*].

The process for solving boundary value problems numerically is outlined below using the example:

$$\frac{d^2u}{dx^2} - 12xu = -1, \quad 0 < x < 1$$
$$u(0) = 1, \qquad u(1) = -1$$

Step 1 – Preliminary Work/Hand Calculations

- Write out the generic i^{th} replacement equation, keeping Δx as a variable.

  EX: $\dfrac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - 12x_i u_i = -1$ for $i = 1,2,\ldots,n-1$

- Solve this equation for u_i. **EX**: $u_i = \dfrac{(\Delta x)^2 + u_{i+1} + u_{i-1}}{2 + 12x_i(\Delta x)^2}$ for $i = 1,2,\ldots,n-1$

  This is the equation that will be iterated starting with initial guesses for u_{i+1} and u_{i-1} (since they are also unknowns at this point).

Step 2 – Preliminary Work/Spreadsheet Setup

- Open a new Excel Spreadsheet. Click on the  button and then click on the Excel Options button at the bottom. In the window that opens select the Formulas . Under Calculation Options, select Manual. Check the Enable iterative calculation box and set Maximum Iterations to 25 and Maximum Change to 0.001. Click OK.

  The spreadsheet is now in manual mode, which means that it will only recalculate values when you press F9. So you must press F9 anytime you need the spreadsheet to recalculate values, e.g. after you fill in a row or column with a formula.

- Enter the labels into the spreadsheet as shown below:

	A	B	C	D	E	F	G	H
1	*grid pt i:*		0	1	2	3	4	5
2								
3	xi							
4	ui							
5	u exact							
6								
7	*error*							
8								
9	parameters							
10	a=							
11	b=							
12	n=							
13	dx=							
14								

Step 3 –Spreadsheet Setup(for specific equation)

- Enter the values for *a* and *b*, the left and right endpoints, into cells B10 and B11. **EX**: a = 0, b = 1
- Enter the value for n, the number of subintervals, into cell B12. **EX**: n = 5

[*] Recall from Calculus I, Newton's method was an iterative method for finding roots of equations.

- Enter the formula to compute Δx into cell B13. **EX**: Type =(B11-B10)/B12 into cell B13.
- To get the values of the grid points x_i,
  - Type =B10 into cell C3.
  - Type =C3+B13 into cell D3.
    [Note: The $ makes it an absolute reference, i.e. it will not change when you fill the formula into other cells.]
  - Fill this formula into the remaining cells in row 3. **EX**: Fill cells D3 through H3 since n = 5
    Note that Excel did not compute the new values when you filled the formula. **Press F9** to have Excel recalculate the formulas. You should now see the values from 0 to 1 in increments of 0.2 in row 3.
- Enter the values of the boundary conditions u_0 and u_n into the corresponding cells in row 4. [**Note**: If the boundary conditions involve derivatives, this step must be modified. – See additional instructions]
  EX: Enter 1 into cell C4 and -1 into cell H4.
- Enter the formula for u_1 found in the preliminary work into cell D4, substituting the cell references C4 for u_0, E4 for u_2, D3 for x_1 and B13 for Δx.

  EX: $u_1 = \dfrac{(\Delta x)^2 + u_2 + u_0}{2 + 12x_1(\Delta x)^2}$ corresponds to typing =(B13^2+E4+C4)/(2+12*D3*B13^2) into cell D4.
- Fill this formula into the remaining cells in row 4.
  EX: Fill cells D4 through G4 since n = 5 (H4 already has boundary condition) **Press F9** to recalculate.

If you can solve the Boundary Value Problem by hand to get an exact solution, complete the following steps. Otherwise, skip to Step 4.
Note: The exact solution for our example contains Airy Functions, which are not predefined functions in Excel. Therefore, these steps will not work on our example although it has an exact solution. (I did find the exact solution values in Maple and copied them to Excel for comparison in the example spreadsheets)

- Solve the BVP by hand to obtain the exact solution $u(x)$.
- Enter the formula for $u(x)$ into cell C5, substituting the cell reference C3 for x. Fill this formula into the remaining cells of row 5.
- Type =abs(C4-C5) into cell C7 to compute the absolute value of the error between the exact and approximate value.

Step 4 – Iteration and Graphing

- Highlight all of the cells in rows 4 through 7 and change the decimal default to be 4 decimal places.
- Press F9 repeatedly until the solution values no longer change.
- Highlight rows 3 and 4 (and 5, if you have computed the exact solution) and create a scatter plot using the Chart Wizard.

Important!

Before you close Excel, make sure you change the Calculation Mode back to Automatic and uncheck the Enable iterative calculations box. You may also want to change the Maximum iterations back to 100. This step is especially important when working on your own computer or in the computer lab if you do not restart your computer.

Homework: Section 7.1, p. 402: #1, 3, 7, 10 [Use iteration to find a numerical solution for n = 4, 16, and 100. When possible, compare your solution to the exact solution and graph both solutions.]