1. (a). Evaluate $\int \cos^3 x \sin x \, dx$

(b). Why can't you evaluate the integral $\int \cos^3 x \, dx$?

(c). So we can integrate integrals of the forms

$\int \sin^m x \cos x \, dx$ using the substitution $u =$ ___$\sin x$___ since the $du =$ ___$\cos x \, dx$___ appears and the integral becomes: ___$\int u^m \, du$___

$\int \cos^n x \sin x \, dx$ using the substitution $u =$ ___$\cos x$___ since the $du =$ ___$-\sin x \, dx$___ appears and the integral becomes: ___$-\int u^n \, du$___

2. Back to $\int \cos^3 x \, dx$.

We would like ***one*** extra factor of ___$\sin x$ or $\cos x$___ .

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$$

So let's take one out. But for this to work, we must have $u =$ ___$\sin x$___ to get $du = \cos x \, dx$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

How can we convert the remaining $\cos^2 x$ into sines?

Use the identity $\cos^2 x = 1 - \sin^2 x$

$$= \int 1 - u^2 \, du$$

Make the u-substitution and viola:

$$= u - \frac{1}{3}u^3 + C \quad = \sin x - \frac{1}{3}\sin^3 x + C$$

3. Let's try another one:

$$\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cos^2 x \sin x \, dx$$

Should we take out extra $\sin x$ or $\cos x$?

$$= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx$$

Convert everything else into ___cosines___

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= -\int \left(1 - u^2\right)^2 \cdot u^2 \, du$$

Make the u-substitution and expand:

$$= -\int \left(1 - 2u^2 + u^4\right) \cdot u^2 \, du$$

$$= \int -u^2 + 2u^4 - u^6 \, du$$

$$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C \quad = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

What if both $\sin x$ and $\cos x$ are raised to even powers? Use the half-angle identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

4.

$$\int_0^{\pi/2} \sin^4 x\, dx = \int_0^{\pi/2} \left(\sin^2 x\right)^2 \, dx$$ Need $\sin^2 x$ to use identity

$$= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 \, dx$$ Use Identity

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\, dx$$ Expand. But $\cos^2 2x$ is a problem.

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x \right) dx$$ Use Identity again

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\, dx = \int_0^{\pi/2} \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\, dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\Big|_0^{\pi/2} = \frac{3}{8}\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right) + \frac{1}{32}\sin\left(4\cdot\frac{\pi}{2}\right) - 0 = \frac{3\pi}{16}$$

Strategy for $\int \sin^m x \cos^n x\, dx$

(a). If power of cosine is odd, i.e. $n = 2k+1$, factor out single factor of cosine and convert the rest to sine using the identity $\cos^2 x = 1 - \sin^2 x$. Then use $u =$ $\sin x$ and $du =$ $\cos x\, dx$:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x\, dx &= \int \sin^m x \cos^{2k} x \cos x\, dx = \int \sin^m x \left(\cos^2 x\right)^k \cos x\, dx \\ &= \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x\, dx \qquad \text{Let } u = \sin x \Rightarrow du = \cos x\, dx \\ &= \int u^m \left(1 - u^2\right)^k du\end{aligned}$$

(b). If power of sine is odd, i.e. $m = 2k+1$, factor out single factor of sine and convert the rest to cosine using the identity $\sin^2 x = 1 - \cos^2 x$. Then use $u =$ $\cos x$ and $du =$ $-\sin x\, dx$:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x\, dx &= \int \sin^{2k} x \cos^n x \sin x\, dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x\, dx \\ &= \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x\, dx \qquad \text{Let } u = \cos x \Rightarrow du = -\sin x\, dx \\ &= -\int \left(1 - u^2\right)^k u^n\, du\end{aligned}$$

(c). If both powers are even, use 1/2 angle identities:

$$\begin{aligned}\sin^2 x = \frac{1}{2}(1 - \cos 2x) &= \frac{1}{2} - \frac{1}{2}\cos 2x \\ \cos^2 x = \frac{1}{2}(1 + \cos 2x) &= \frac{1}{2} + \frac{1}{2}\cos 2x\end{aligned}$$

For $\int \tan^m x \sec^n x \, dx$:

If you want $u = \tan x$, you need $du =$ ___$\sec^2 x \, dx$___ to appear.

If you want $u = \sec x$, you need $du =$ ___$\sec x \tan x \, dx$___ to appear.

Strategy for $\int \tan^m x \sec^n x \, dx$

(a). If power of secant is even, i.e. $n = 2k$, factor out $\sec^2 x$ and convert the rest to tangents using the identity $\sec^2 x = 1 + \tan^2 x$. Then use $u =$ ___$\tan x$___ and $du =$ ___$\sec^2 x \, dx$___:

$$\begin{aligned}\int \tan^m x \sec^{2k} x \, dx &= \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx \qquad \text{Let } u = \tan x \Rightarrow du = \sec^2 x \, dx \\ &= \int u^m (1 + u^2)^{k-1} \, du\end{aligned}$$

(b). If power of tangent is odd, i.e. $m = 2k + 1$, factor out $\sec x \tan x$ and convert the rest to secants using the identity $\tan^2 x = \sec^2 x - 1$. Then use $u =$ ___$\sec x$___ and $du =$ ___$\sec x \tan x \, dx$___:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \, dx &= \int \tan^{2k} x \sec^{n-1} x \; \sec x \tan x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \; \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \; \sec x \tan x \, dx \qquad \text{Let } u = \sec x \Rightarrow du = \sec x \tan x \, dx \\ &= \int (u^2 - 1)^k u^{n-1} \, du\end{aligned}$$

(c). If neither of the above case applies, try something else (other identities, IBP, etc.)

5. $\int \tan^4 x \sec^6 x \, dx$

6. $\int \tan^3 x \sec^3 x \, dx$

Trig Identities and Formulas You Already Know

Helpful Trig. Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$\begin{aligned}
\sin A \cos B &= \tfrac{1}{2}[\sin(A+B) + \sin(A-B)] &= \tfrac{1}{2}\sin(A+B) + \tfrac{1}{2}\sin(A-B) \\
\cos A \cos B &= \tfrac{1}{2}[\cos(A+B) + \cos(A-B)] &= \tfrac{1}{2}\cos(A+B) + \tfrac{1}{2}\cos(A-B) \\
\sin A \sin B &= \tfrac{1}{2}[\cos(A-B) - \cos(A+B)] &= \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B)
\end{aligned}$$

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

Useful u-substitutions

$u = \sin x$
$du = \cos x\ dx$

$u = \cos x$
$du = -\sin x\ dx$

$u = \tan x$
$du = \sec^2 x\ dx$

$u = \sec x$
$du = \sec x \tan x\ dx$

Integration Formula Shortcuts

$$\int \sin ax\ dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\ dx = \frac{1}{a}\sin ax$$

$$\int \tan x\ dx = \ln|\sec x| \qquad \int \sec x\ dx = \ln|\sec x + \tan x|$$