The Test for Divergence

If $\lim a_n \neq 0$, then the series $\sum a_n$ diverges by Test for Divergence.

Note

If $\lim a_n = 0$, not enough info yet $\Rightarrow$ More Work! [Use another test to determine whether $\sum a_n$ converges or diverges.]

The p-Series Test

The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges for $p \leq 1$.

The Geometric Series Test

The series $\sum ar^n$ converges if $|r| < 1$ and diverges for $|r| \geq 1$. [Use algebra to write in this form.]

Steps for using the Limit Comparison Test (for series with positive terms)

1. Let b_n be the leading order terms of a_n (w/o coefficients). Determine whether $\sum b_n$ converges or diverges.

2. Compute $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$ and verify that it is *nonzero* and *finite*.

3. **Then since $\sum b_n$ is a converging/(or diverging) <u>p-, geom., etc.</u> series, the series $\sum a_n$ converges/(or diverges) by the Limit Comparison Test.**

Steps for using the Comparison Test (for series with positive terms)

1. Look at leading order terms of a_n to find b_n. Determine whether $\sum b_n$ converges or diverges.
[May need to consider bounds on sine or cosine.]

2. Establish the correct inequality:

(a). If $\sum b_n$ converges, need $0 \leq a_n \leq b_n$

(b). If $\sum b_n$ diverges, need $0 \leq b_n \leq a_n$

[May need to adjust b_n by multiplying/dividing by a number to get $\leq$ in the right place.]

3. **Since $\sum b_n$ is a converging/(or diverging) <u>p-, geom., etc.</u> series and $a_n \leq b_n$ /(or $b_n \leq a_n$), the series $\sum a_n$ converges/(or diverges) by the Comparison Test.**

Steps for using the Alternating Series Test

For series of the form: $\sum a_n = \sum (-1)^n b_n$

1. Compute $\lim\limits_{n\to\infty} b_n$

2. Verify that b_n's are (eventually) positive and decreasing. [If not, must use another test]

3. **Since b_n is positive and decreasing and $\lim b_n = 0$, the series $\sum a_n$ converges by the Alternating Series Test.**

Note: **If $\lim b_n \neq 0$, the series $\sum a_n$ diverges by Test for Divergence.**

STEPS FOR USING THE RATIO TEST

1. Compute $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

2. **Since $L < 1$/ (or $L > 1$), the series $\sum a_n$ converges absolutely/(or diverges) by the Ratio Test.**

Note: If $L = 1$, the test fails $\Rightarrow$ must use another test.

STEPS FOR USING THE ROOT TEST

1. Compute $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$

2. **Since $L < 1$/ (or $L > 1$), the series $\sum a_n$ converges absolutely/(or diverges) by the Root Test.**

Note: If $L = 1$, the test fails $\Rightarrow$ must use another test.

STEPS FOR USING THE INTEGRAL TEST

1. Verify that $a_n = f(n)$ is (eventually) continuous, positive, and decreasing.

2. Evaluate the improper integral $\int_1^\infty f(x)\,dx = \begin{cases} \text{finite number} & \Rightarrow \textit{converges} \\ \infty & \Rightarrow \textit{diverges} \end{cases}$

3. **Since the integral converges/(or diverges), the series $\sum a_n$ converges/(or diverges) by the Integral Test.**

STEPS FOR DETERMINING ABSOLUTE AND CONDITIONAL CONVERGENCE

0. If possible, use the Root or Ratio Test and get the result "for free".

1. If those tests are not suitable, use any other test for the series $\sum |a_n|$.

(a). **If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.**

(b). If $\sum |a_n|$ diverges, then no info ***yet*** about $\sum a_n \Rightarrow$ Use any test for the original series $\sum a_n$.

(i). If $\sum a_n$ converges (& $\sum |a_n|$ diverges), then $\sum a_n$ converges conditionally.

(ii). If $\sum a_n$ diverges (& $\sum |a_n|$ diverges), then $\sum a_n$ diverges.