In the following problems, you will try to approximate functions f(x) using polynomial p(x). Complete following by writing the $1^{st} - 5^{th}$ derivatives of the given (general) polynomial. Then evaluate the polynomial at x = 0. The c_n 's are just constants.

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots + c_n x^n \qquad p(0) = c_0$$

$$p'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots + nc_n x^{n-1} \qquad p'(0) =$$

$$p''(x) = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots + n(n-1)c_n x^{n-2} \qquad p''(0) =$$

$$p'''(x) = 6c_3 + 24c_4 x + 60c_5 x^2 + \dots + n(n-1)(n-2)c_n x^{n-3} \qquad p'''(0) =$$

$$p^{(iv)}(x) = 24c_4 + 120c_5 x + \dots + n(n-1)(n-2)(n-3)c_n x^{n-4} \qquad p^{(iv)}(0) =$$

$$p^{(v)}(x) = 120c_5 + 720c_6 x + \dots + n(n-1)(n-2)(n-3)(n-4)c_n x^{n-5} \qquad p^{(v)}(0) =$$

Use the results of polynomial and its derivatives evaluated at x = 0 for the next 2 problems.

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1. Given the function $f(x) = \frac{1}{1-x} = (1-x)^{-1}$, find the derivatives and evaluate the function at x = 0. From the top of page 1, also include the general polynomial evaluated at x = 0.

$$f(x) = (1-x)^{-1} = \frac{1}{1-x} \qquad f(0) = 1 \qquad p(0) = c_0$$

$$f'(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2} \qquad = \frac{1}{(1-x)^2} \qquad f'(0) = \qquad p'(0) = 0$$

$$f''(x) = -2(1-x)^{-3} \cdot (-1) = 2(1-x)^{-3} = \frac{2}{(1-x)^3} \qquad f''(0) = \qquad p''(0) =$$

$$f'''(x) = -6(1-x)^{-4} \cdot (-1) = 6(1-x)^{-4} = \frac{6}{(1-x)^4} \qquad f'''(0) = \qquad p'''(0) =$$

$$f^{(iv)}(x) = -24(1-x)^{-5} \cdot (-1) = 24(1-x)^{-5} = \frac{24}{(1-x)^5} \qquad f^{(iv)}(0) = p^{(iv)}(0) =$$

$$f^{(v)}(x) = -120(1-x)^{-6} \cdot (-1) = 120(1-x)^{-5} = \frac{120}{(1-x)^5}$$

- $f^{(v)}(0) = p^{(v)}(0) =$
- (a). Find c_0 so that p(x) and f(x) match at x = 0 i.e. Set f(0) = p(0)

Find c_2 so that the second derivatives match at x = 0i.e. Set f''(0) = p''(0)

Find c_4 so that the fourth derivatives match at x = 0i.e. Set $f^{(iv)}(0) = p^{(iv)}(0)$ Find c_1 so that the first derivatives match at x = 0i.e. Set f'(0) = p'(0)

Find c_3 so that the third derivatives match at x = 0i.e. Set f'''(0) = p'''(0)

Find c_5 so that the fifth derivatives match at x = 0i.e. Set $f^{(v)}(0) = p^{(v)}(0)$

(b). Use the coefficients found in part (a) to write down the resulting polynomial up to x^5 . i.e Write down $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$ since we're claiming $f(x) \approx p(x)$.

$$\frac{1}{1-x} \approx \qquad \qquad \longleftarrow \text{ fill in polynomial}$$

(c). What do you notice about all of the coefficients c_i so far? Use this observation to write down an infinite (power) series that will match the function up to all (infinitely many) derivatives.

$$\frac{1}{1-x} = \sum_{n=1}^{\infty}$$

(d). Replace the x with an r in the above equation:

$$\frac{1}{1-r} =$$

This series and formula should look familiar – what is it?

For what values of r (and ultimately x) does it converge?

(e). Graph the function $f(x) = \frac{1}{1-x}$ and the polynomial p(x) from part (b) on the same screen. Use the viewing window $[-3,3] \times [-3,3]$ and sketch the polynomial on the graph below.

From the graph, How well does the polynomial approximate f(x) near x = 0? How well does it approximate f(x) for |x| < 1? How well does it approximate f(x) for |x| > 1?



2. Repeat the process for the function $f(x) = \sin x$.

From the top of page 1, also include the general polynomial evaluated at x = 0.

f(x)	=	$\sin x$	f(0)	=	p(0)	=	c_0
f'(x)	=		f'(0)	=	p'(0)	=	c_1
$f^{\prime\prime}(x)$	=		f''(0)	=	$p^{\prime\prime}(0)$	=	$2c_2$
$f^{\prime\prime\prime}(x)$	=		$f^{\prime\prime\prime}(0)$	=	$p^{\prime\prime\prime}(0)$	=	$6c_3$
$f^{(iv)}(x)$	=		$f^{(iv)}(0)$	=	$p^{(iv)}(0)$	=	$24c_4$
$f^{(v)}(x)$	=		$f^{(v)}(0)$	=	$p^{(v)}(0)$	=	$120c_{5}$

- (a). Using the above information, find the values of the coefficients c_0, c_1, c_2, c_3, c_4 , and c_5 so that that f(x) and p(x) match up to their 5th derivatives.
- (b). Write down the resulting polynomial up to x^5 and simplify. i.e Write down $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5$ since we're claiming $f(x) \approx p(x)$.

 $\sin x \approx$

(c). Graph $f(x) = \sin x$ and the polynomial p(x) from part (b) on your Calculator. Use the viewing window $-\pi \le x \le \pi$ and $-3 \le y \le 3$.

Sketch the polynomial onto the sine curve below.



How good does the polynomial approximate the sine function near x = 0?

 \leftarrow fill in polynomial