Name: Key

Math 152, Calculus II – Crawford

Exam 2

07 April 2015

- No calculators, books, or notes (in any form) allowed.
- Clearly indicate your answers.
- ***Show all your work*** – partial credit may be given for written work.
- Evaluate trigonometric, exponential, and logarithmic expressions for standard values.
- Good Luck!

Score

1	/40
2	/10
3	/8
4	/20
5	/20
6	/6
Total	/100

Formulas that you may or may not find helpful

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ $\qquad$ $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$ $\qquad$ $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)] = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)] = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

$\sin(-\theta) = -\sin\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$ $\qquad$ $\int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta|$

$$\frac{1}{2}\Delta x\,[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$\frac{1}{3}\Delta x\,[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

1. (40 pts). Evaluate the following integrals. Show all your work.

(a). $\int e^{3x} \sin(x)\, dx$

	(diff) u	(int) dv
$+$	e^{3x}	$\sin x$
$-$	$3e^{3x}$	$-\cos x$
$+$	$9e^{3x}$	$-\sin x$

$$\int e^{3x} \sin x\, dx = -e^{3x}\cos x + 3e^{3x}\sin x - 9\int e^{3x}\sin x\, dx$$

Add to both sides: $9\int e^{3x} \sin x\, dx \qquad 9\int e^{3x}\sin x\, dx$

$$10\int e^{3x}\sin x\, dx = -e^{3x}\cos x + 3e^{3x}\sin x$$

Divide

$$\Rightarrow \int e^{3x}\sin x\, dx = \boxed{-\tfrac{1}{10}e^{3x}\cos x + \tfrac{3}{10}e^{3x}\sin x + C}$$

(b). $\int \frac{1}{x^2\sqrt{x^2+9}}\, dx$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$$\int \frac{1}{9\tan^2\theta\sqrt{9\tan^2\theta+9}} \cdot 3\sec^2\theta\, d\theta = \int \frac{\sec^2\theta}{3\tan^2\theta\sqrt{9(\tan^2\theta+1)}}\, d\theta$$

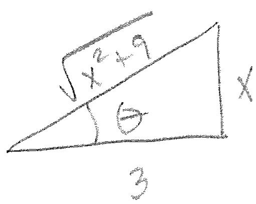
$$= \int \frac{\sec^2\theta}{3\tan^2\theta\sqrt{9\sec^2\theta}}\, d\theta = \int \frac{\sec^{\cancel{2}}\theta}{3\tan^2\theta\cdot 3\cancel{\sec\theta}}\, d\theta = \int \frac{\sec\theta}{9\tan^2\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}\, d\theta = \frac{1}{9}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{9}\int \frac{1}{u^2}\, du = \frac{1}{9}\left(-\frac{1}{u}\right) + C = -\frac{1}{9}\cdot\frac{1}{\sin\theta} + C = -\frac{1}{9}\csc\theta + C$$



$\tan\theta = \frac{x}{3}$

$h^2 = x^2 + 3^2$

$$= \boxed{-\frac{1}{9}\,\frac{\sqrt{x^2+9}}{x} + C}$$

(c). $\int \sin^4(2x)\cos^3(2x)\,dx$ $= \int \sin^4(2x)\cdot\cos^2(2x)\underbrace{\cos(2x)\,dx}_{\text{want as } du}$

$= \int \sin^4(2x)(1-\sin^2(2x))\cos(2x)dx$

$u = \sin(2x)$

$du = 2\cos(2x)dx$

$\frac{1}{2}du = \cos(2x)dx$

$= \frac{1}{2}\int u^4(1-u^2)\,du$

$= \frac{1}{2}\int u^4 - u^6\,du$

$= \frac{1}{2}\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right] + C = \frac{1}{10}u^5 - \frac{1}{14}u^7 + C$

$= \boxed{\frac{1}{10}\sin^5(2x) - \frac{1}{14}\sin^7(2x) + C}$

(d). $\int \frac{x-1}{x^2(x+1)}\,dx$

$\frac{x-1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ LCD $= x^2(x+1)$

$x-1 = Ax(x+1) + B(x+1) + Cx^2$

$x-1 = Ax^2 + Ax + Bx + B + Cx^2$

$x-1 = (A+C)x^2 + (A+B)x + B$

LHS = RHS

x^2: $0 = A + C$ $0 = 2 + C \Rightarrow C = -2$

x: $1 = A + B$ $1 = A - 1 \Rightarrow A = 2$

const: $-1 = B$

$\int \frac{x-1}{x^2(x+1)}\,dx = \int \frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+1}\,dx$

$= \boxed{2\ln|x| + \frac{1}{x} - 2\ln|x+1| + C}$

2. (10 pts). Find the area bounded by $f(x) = x^2 \ln x$ on the interval $[1, e]$. [Simplify your answer.]

$$\int_1^e x^2 \ln x \, dx$$

$$u = \ln x \qquad dv = x^2$$
$$du = \frac{1}{x}\,dx \qquad v = \frac{1}{3}x^3$$

$$= \frac{1}{3}x^3 \ln x - \int_1^e \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int_1^e x^2\,dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\cdot\frac{1}{3}x^3 \Big|_1^e$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 \Big|_1^e$$

$$= \frac{1}{3}e^3 \overset{1}{\ln e} - \frac{1}{9}e^3 - \Big(\frac{1}{3}\underbrace{\ln 1}_{0} - \frac{1}{9}\Big)$$

$$= \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9}$$

$$= \boxed{\frac{2}{9}e^3 + \frac{1}{9}}$$

3. (8 pts). Use Simpson's Rule with $n = 6$ to approximate the integral $\int_2^5 \frac{1}{\ln x}\,dx$. [Do not simplify!!]

2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
x_0	x_1	x_2	x_3	x_4	x_5	x_6

$$\Delta x = \frac{5-2}{6} = \frac{3}{6} = \frac{1}{2}$$

$$S_6 = \frac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$$

$$= \frac{1}{3}\cdot\frac{1}{2}\left[f(2) + 4f(\tfrac{5}{2}) + 2f(3) + 4f(\tfrac{7}{2}) + 2f(4) + 4f(\tfrac{9}{2}) + f(5)\right]$$

$$= \boxed{\frac{1}{6}\left[\frac{1}{\ln 2} + 4\cdot\frac{1}{\ln(5/2)} + 2\cdot\frac{1}{\ln 3} + 4\cdot\frac{1}{\ln(7/2)} + 2\cdot\frac{1}{\ln 4} + 4\cdot\frac{1}{\ln(9/2)} + \frac{1}{\ln 5}\right]}$$

4. (20 pts). Evaluate the following integrals or show that it is a divergent improper integral.

(a). $\int_4^8 \frac{1}{\sqrt{x-4}}\,dx = \lim_{t\to 4^+} \int_t^8 \frac{1}{\sqrt{x-4}}\,dx$

$u = x-4$
$du = dx$

$= \lim_{t\to 4^+} \int_{x=t}^{x=8} \frac{1}{\sqrt{u}}\,du = \lim_{t\to 4^+} \int_{x=t}^{x=8} u^{-1/2}\,du$

$= \lim_{t\to 4^+} 2\cdot u^{1/2}\Big|_{x=t}^{x=8} = \lim_{t\to 4^+} 2\sqrt{x-4}\Big|_t^8$

$= \lim_{t\to 4^+} 2\sqrt{8-4} - 2\sqrt{t-4} = 2\sqrt{4} - 2(0) = 2\cdot 2 = \boxed{4}$

(ie converges to 4)

(b). $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx = \int_{-\infty}^{0} x^3 e^{-x^4}\,dx + \int_0^{\infty} x^3 e^{-x^4}\,dx$

① ②

① $\int_{-\infty}^{0} x^3 e^{-x^4}\,dx = \lim_{t\to -\infty} \int_t^0 x^3 e^{-x^4}\,dx$

$u = -x^4$
$du = -4x^3\,dx$
$-\frac{1}{4}du = x^3\,dx$

$= \lim_{t\to -\infty} \int_{x=t}^{x=0} -\frac{1}{4} e^u\,du$

$= \lim_{t\to -\infty} -\frac{1}{4} e^u \Big|_{x=t}^{x=0} = \lim_{t\to \infty} -\frac{1}{4} e^{-x^4}\Big|_t^0 = \lim_{t\to -\infty} -\frac{1}{4}e^0 + \frac{1}{4}e^{-t^4}$

$0 \quad (e^{-\infty}$

① $= -\frac{1}{4}$

② $\int_0^{\infty} x^3 e^{-x^4}\,dx = \lim_{t\to\infty} \int_0^t x^3 e^{-x^4}\,dx$

Same u-subs & same work leads to

$= \lim_{t\to\infty} -\frac{1}{4}e^{-x^4}\Big|_0^t = \lim_{t\to\infty} -\frac{1}{4}e^{-t^4} + \frac{1}{4}e^0 = \frac{1}{4} = ②$

$0 \; (e^{-\infty})$

So $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx$ converges to $-\frac{1}{4} + \frac{1}{4} = \boxed{0}$

5. (20 pts). Determine whether the following sequences converge or diverge. **If it converges, find the limit. If it diverges, clearly explain the reason why.** [Clearly indicate $+\infty$ or $-\infty$ in the case of an infinite limit.]

(a). $a_n = \dfrac{3n^2 + 2n - 1}{2 - 4n^2}$

$= \boxed{\dfrac{3}{-4}}$

(b). $a_n = 3 - (0.4)^n$

$= 3 - 0$

$= \boxed{3}$

Since $r = 0.4$ and $|r| < 1$ then $r^n \to 0$

(c). $a_n = \dfrac{\sin n}{1 + n^2}$

Know: $-1 \le \sin n \le 1$

Divide: $\dfrac{-1}{1+n^2} \le \dfrac{\sin n}{1+n^2} \le \dfrac{1}{1+n^2}$

Take limit; $\lim\limits_{n\to\infty} \dfrac{-1}{1+n^2} \le \lim\limits_{n\to\infty} \dfrac{\sin n}{1+n^2} \le \lim\limits_{n\to\infty} \dfrac{1}{1+n^2}$

$0 \le \lim\limits_{n\to\infty} \dfrac{\sin n}{1+n^2} \le 0$

$\therefore \lim\limits_{n\to\infty} \dfrac{\sin n}{1+n^2} = \boxed{0}$ by the Squeeze Theorem.

6. (6 pts). *True or False.* Determine whether the following statements are true or false.

T F If $f(x) \le g(x)$ and $\int_0^\infty f(x)\,dx$ converges, then $\int_0^\infty g(x)\,dx$ also converges.

It would be true if: $g(x) \le f(x)$

T F The fraction $\dfrac{2x^2 + x - 3}{x(2x+3)^3(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{2x+3} + \underbrace{\dfrac{C}{(2x+3)^2} + \dfrac{D}{(2x+3)^3}}_{\text{corrected to be true}} + \dfrac{Ex+F}{x^2+1}$.

T F $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{u}{u^2-1}\,du$ using the rationalizing substitution $u = \sqrt{x+1}$.

$u^2 = x+1 \Rightarrow x = u^2 - 1$

$dx = 2u\,du$ missing

ie $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{u}{u^2-1} \cdot 2u\,du$ is correct.