

1. (a). Evaluate $\int \cos^3 x \sin x \, dx$

(b). Why can't you evaluate the integral $\int \cos^3 x \, dx$?

(c). So we can integrate integrals of the forms

$\int \sin^m x \cos x \, dx$ using the substitution $u = \underline{\sin x}$ since the $du = \underline{\cos x \, dx}$ appears and the integral becomes: $\underline{\int u^m \, du}$

$\int \cos^n x \sin x \, dx$ using the substitution $u = \underline{\cos x}$ since the $du = \underline{-\sin x \, dx}$ appears and the integral becomes: $\underline{-\int u^n \, du}$

2. Back to $\int \cos^3 x \, dx$.

We would like one extra factor of $\sin x$ or $\cos x$.

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx$$

So let's take one out. But for this to work,

we must have $u = \underline{\sin x}$ to get $du = \cos x \, dx$

How can we convert the remaining $\cos^2 x$ into sines?

Use the identity $\cos^2 x = 1 - \sin^2 x$

Make the u -substitution and viola:

$$= \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int 1 - u^2 \, du$$

$$= u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C$$

3. Let's try another one:

$$\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cos^2 x \sin x \, dx$$

Should we take out extra $\sin x$ or $\cos x$?

$$= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx$$

Convert everything else into cosines

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx$$

$$= -\int (1 - u^2)^2 \cdot u^2 \, du$$

Make the u -substitution and expand:

$$= -\int (1 - 2u^2 + u^4) \cdot u^2 \, du$$

$$= \int -u^2 + 2u^4 - u^6 \, du$$

$$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

What if both $\sin x$ and $\cos x$ are raised to even powers?

Use the half-angle identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

4. $\int_0^{\pi/2} \sin^4 x \, dx = \int_0^{\pi/2} (\sin^2 x)^2 \, dx$ Need $\sin^2 x$ to use identity

$$= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)^2 \, dx$$
 Use Identity

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x \, dx$$
 Expand. But $\cos^2 2x$ is a problem.

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos 4x \right) \, dx$$
 Use Identity again

$$= \int_0^{\pi/2} \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{8} + \frac{1}{8} \cos 4x \, dx = \int_0^{\pi/2} \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \, dx$$

$$= \left. \frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x \right|_0^{\pi/2} = \frac{3}{8} \left(\frac{\pi}{2} \right) - \frac{1}{4} \sin \left(2 \cdot \frac{\pi}{2} \right) + \frac{1}{32} \sin \left(4 \cdot \frac{\pi}{2} \right) - 0 = \frac{3\pi}{16}$$

Strategy for $\int \sin^m x \cos^n x \, dx$

(a). If power of cosine is odd, i.e. $n = 2k + 1$, factor out single factor of cosine and convert the rest to sine using the identity $\cos^2 x = 1 - \sin^2 x$. Then use $u = \sin x$ and $du = \cos x \, dx$:

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x \cos^{2k} x \cos x \, dx = \int \sin^m x (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx && \text{Let } u = \sin x \Rightarrow du = \cos x \, dx \\ &= \int u^m (1 - u^2)^k \, du \end{aligned}$$

(b). If power of sine is odd, i.e. $m = 2k + 1$, factor out single factor of sine and convert the rest to cosine using the identity $\sin^2 x = 1 - \cos^2 x$. Then use $u = \cos x$ and $du = -\sin x \, dx$:

$$\begin{aligned} \int \sin^{2k+1} x \cos^n x \, dx &= \int \sin^{2k} x \cos^n x \sin x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx && \text{Let } u = \cos x \Rightarrow du = -\sin x \, dx \\ &= - \int (1 - u^2)^k u^n \, du \end{aligned}$$

(c). If both powers are even, use $1/2$ angle identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2} + \frac{1}{2} \cos 2x$$

For $\int \tan^m x \sec^n x \, dx$:

If you want $u = \tan x$, you need $du = \underline{\sec^2 x \, dx}$ to appear.

If you want $u = \sec x$, you need $du = \underline{\sec x \tan x \, dx}$ to appear.

Strategy for $\int \tan^m x \sec^n x \, dx$

- (a). If power of secant is even, i.e. $n = 2k$, factor out $\sec^2 x$ and convert the rest to tangents using the identity $\sec^2 x = 1 + \tan^2 x$. Then use $u = \underline{\tan x}$ and $du = \underline{\sec^2 x \, dx}$:

$$\begin{aligned} \int \tan^m x \sec^{2k} x \, dx &= \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx = \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx \\ &= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx && \text{Let } u = \tan x \Rightarrow du = \sec^2 x \, dx \\ &= \int u^m (1 + u^2)^{k-1} \, du \end{aligned}$$

- (b). If power of tangent is odd, i.e. $m = 2k + 1$, factor out $\sec x \tan x$ and convert the rest to secants using the identity $\tan^2 x = \sec^2 x - 1$. Then use $u = \underline{\sec x}$ and $du = \underline{\sec x \tan x \, dx}$:

$$\begin{aligned} \int \tan^{2k+1} x \sec^n x \, dx &= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx && \text{Let } u = \sec x \Rightarrow du = \sec x \tan x \, dx \\ &= \int (u^2 - 1)^k u^{n-1} \, du \end{aligned}$$

- (c). If neither of the above case applies, try something else (other identities, IBP, etc.)

5. $\int \tan^4 x \sec^6 x \, dx$

6. $\int \tan^3 x \sec^3 x \, dx$

Trig Identities and Formulas You Already Know

Helpful Trig. Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)] = \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)] = \frac{1}{2} \cos(A + B) + \frac{1}{2} \cos(A - B)$$

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B)$$

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

Useful u -substitutions

$$\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}$$

$$\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned}$$

$$\begin{aligned} u &= \tan x \\ du &= \sec^2 x \, dx \end{aligned}$$

$$\begin{aligned} u &= \sec x \\ du &= \sec x \tan x \, dx \end{aligned}$$

Integration Formula Shortcuts

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax$$

$$\int \tan x \, dx = \ln |\sec x|$$

$$\int \sec x \, dx = \ln |\sec x + \tan x|$$