

$$1. \sum_{n=1}^{\infty} \frac{-1}{n^2 + 7n + 12}$$

Note: $\lim a_n = \underline{0}$ and leading order $\underline{\frac{1}{n^2}}$

Rewrite $\frac{-1}{n^2 + 7n + 12}$ in partial fraction decomposition

(i.e. go backwards from common denominator)

(a). Factor denominator $\frac{-1}{n^2 + 7n + 12} = \frac{-1}{(n+3)(n+4)}$

(b). "Break up" fraction: $\frac{-1}{(n+3)(n+4)} = \frac{A}{n+3} + \frac{B}{n+4}$

Need to determine A and B .

(c). Recombine with common denominator: $\frac{-1}{(n+3)(n+4)} = \frac{A(n+4) + B(n+3)}{(n+3)(n+4)}$

Numerators must be equal: $-1 = An + 4A + Bn + 3B = (A+B)n + (4A+3B)$

Collect Terms: LHS = RHS

$$n: \quad 0 = A + B$$

Solve for A and B

$$\text{constant:} \quad -1 = 4A + 3B$$

(d). Then $\frac{-1}{(n+3)(n+4)} = \frac{-1}{n+3} + \frac{1}{n+4}$

So does $\sum_{n=1}^{\infty} \frac{-1}{n^2 + 7n + 12} = \sum_{n=1}^{\infty} \left(\frac{-1}{n+3} + \frac{1}{n+4} \right)$ converge? If so, to what?

Look at partial sums s_k :

$$s_1 = \left(-\frac{1}{4} + \frac{1}{5} \right)$$

$$s_2 = \left(-\frac{1}{4} + \frac{1}{5} \right) + \left(-\frac{1}{5} + \frac{1}{6} \right)$$

$$s_3 = \left(-\frac{1}{4} + \frac{1}{5} \right) + \left(-\frac{1}{5} + \frac{1}{6} \right) + \left(-\frac{1}{6} + \frac{1}{7} \right)$$

$$s_4 = \left(-\frac{1}{4} + \frac{1}{5} \right) + \left(-\frac{1}{5} + \frac{1}{6} \right) + \left(-\frac{1}{6} + \frac{1}{7} \right) + \left(-\frac{1}{7} + \frac{1}{8} \right)$$

$\vdots$

$$s_k = \left(-\frac{1}{4} + \frac{1}{5} \right) + \left(-\frac{1}{5} + \frac{1}{6} \right) + \left(-\frac{1}{6} + \frac{1}{7} \right) + \left(-\frac{1}{7} + \frac{1}{8} \right) + \cdots + \left(-\frac{1}{k+2} + \frac{1}{k+3} \right) + \left(-\frac{1}{k+3} + \frac{1}{k+4} \right)$$

$$s_k = -\frac{1}{4} + \frac{1}{k+4}$$

So $\lim s_k = -\frac{1}{4} + 0 = -\frac{1}{4}$.

Thus the infinite series $\sum_{n=1}^{\infty} \frac{-1}{n^2 + 7n + 12} = -\frac{1}{4}$

since the sequence of partial sums $s_k \Rightarrow -\frac{1}{4}$.

$$2. \sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

$$\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} \implies$$

$$\implies 3 = (A+B)n + 3A$$

Collect Terms: LHS = RHS

$$n: \quad 0 = A + B$$

$$A = 1 \quad B = -1$$

$$\text{constant:} \quad 3 = 3A$$

$$\text{So } \sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$$

$$s_1 = \left(1 - \frac{1}{4}\right)$$

$$s_2 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right)$$

$$s_3 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right)$$

$$s_4 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right)$$

$$s_5 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right)$$

$$s_6 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{9}\right)$$

$$s_7 = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{9}\right) + \left(\frac{1}{7} - \frac{1}{10}\right)$$

$\vdots$

$$s_k = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{9}\right) + \left(\frac{1}{7} - \frac{1}{10}\right) + \dots$$

$$+ \dots + \left(\frac{1}{k-3} - \frac{1}{k}\right) + \left(\frac{1}{k-2} - \frac{1}{k+1}\right) + \left(\frac{1}{k-1} - \frac{1}{k+2}\right) + \left(\frac{1}{k} - \frac{1}{k+3}\right)$$

$$s_k = 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{k+1} - \frac{1}{k+2} - \frac{1}{k+3}$$

$$\text{Then } \lim s_k = 1 + \frac{1}{2} + \frac{1}{3} - 0 - 0 - 0 = \frac{11}{6}$$

$$\text{Therefore the infinite series } \sum_{n=1}^{\infty} \frac{3}{n(n+3)} \text{ converges to } \underline{\underline{\frac{11}{6}}}$$