In the following problems, you will try to approximate functions $f(x)$ using polynomial $p(x)$. Complete following by writing the $1^{\text{st}} - 5^{\text{th}}$ derivatives of the given (general) polynomial. Then evaluate the polynomial at $x = 0$. The c_n's are just constants.

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots + c_nx^n \qquad p(0) = c_0$$

$$p'(x) = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \cdots + nc_nx^{n-1} \qquad p'(0) =$$

$$p''(x) = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots + n(n-1)c_nx^{n-2} \qquad p''(0) =$$

$$p'''(x) = 6c_3 + 24c_4x + 60c_5x^2 + \cdots + n(n-1)(n-2)c_nx^{n-3} \qquad p'''(0) =$$

$$p^{(iv)}(x) = 24c_4 + 120c_5x + \cdots + n(n-1)(n-2)(n-3)c_nx^{n-4} \qquad p^{(iv)}(0) =$$

$$p^{(v)}(x) = 120c_5 + 720c_6x + \cdots + n(n-1)(n-2)(n-3)(n-4)c_nx^{n-5} \qquad p^{(v)}(0) =$$

Use the results of polynomial and its derivatives evaluated at $x = 0$ for the next 2 problems.

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1. Given the function $f(x) = \frac{1}{1-x} = (1-x)^{-1}$, find the derivatives and evaluate the function at $x = 0$. From the top of page 1, also include the general polynomial evaluated at $x = 0$.

$$f(x) = (1-x)^{-1} = \frac{1}{1-x} \qquad f(0) = 1 \qquad p(0) = c_0$$

$$f'(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2} = \frac{1}{(1-x)^2} \qquad f'(0) = \qquad p'(0) =$$

$$f''(x) = -2(1-x)^{-3} \cdot (-1) = 2(1-x)^{-3} = \frac{2}{(1-x)^3} \qquad f''(0) = \qquad p''(0) =$$

$$f'''(x) = -6(1-x)^{-4} \cdot (-1) = 6(1-x)^{-4} = \frac{6}{(1-x)^4} \qquad f'''(0) = \qquad p'''(0) =$$

$$f^{(iv)}(x) = -24(1-x)^{-5} \cdot (-1) = 24(1-x)^{-5} = \frac{24}{(1-x)^5} \qquad f^{(iv)}(0) = \qquad p^{(iv)}(0) =$$

$$f^{(v)}(x) = -120(1-x)^{-6} \cdot (-1) = 120(1-x)^{-5} = \frac{120}{(1-x)^5} \qquad f^{(v)}(0) = \qquad p^{(v)}(0) =$$

(a). Find c_0 so that $p(x)$ and $f(x)$ match at $x = 0$
i.e. Set $f(0) = p(0)$

Find c_1 so that the first derivatives match at $x = 0$
i.e. Set $f'(0) = p'(0)$

Find c_2 so that the second derivatives match at $x = 0$
i.e. Set $f''(0) = p''(0)$

Find c_3 so that the third derivatives match at $x = 0$
i.e. Set $f'''(0) = p'''(0)$

Find c_4 so that the fourth derivatives match at $x = 0$
i.e. Set $f^{(iv)}(0) = p^{(iv)}(0)$

Find c_5 so that the fifth derivatives match at $x = 0$
i.e. Set $f^{(v)}(0) = p^{(v)}(0)$

(b). Use the coefficients found in part (a) to write down the resulting polynomial up to x^5.
i.e Write down $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$ since we're claiming $f(x) \approx p(x)$.

$$\frac{1}{1-x} \approx$$ ⟵ fill in polynomial

(c). What do you notice about all of the coefficients c_i so far?
Use this observation to write down an infinite (power) series that will match the function up to all (infinitely many) derivatives.

$$\frac{1}{1-x} = \sum_{n=}^{\infty}$$

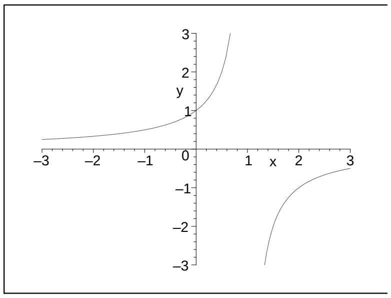
(d). Replace the x with an r in the above equation:

$$\frac{1}{1-r} =$$

This series and formula should look familiar – what is it?

For what values of r (and ultimately x) does it converge?

(e). Graph the function $f(x) = \dfrac{1}{1-x}$ and the polynomial $p(x)$ from part (b) on the same screen.
Use the viewing window $[-3, 3] \times [-3, 3]$ and sketch the polynomial on the graph below.



From the graph,
How well does the polynomial approximate $f(x)$ near $x = 0$?

How well does it approximate $f(x)$ for $|x| < 1$?

How well does it approximate $f(x)$ for $|x| > 1$?

2. Repeat the process for the function $f(x) = \sin x$.
From the top of page 1, also include the general polynomial evaluated at $x = 0$.

$$f(x) = \sin x \qquad f(0) = \qquad p(0) = c_0$$
$$f'(x) = \qquad f'(0) = \qquad p'(0) = c_1$$
$$f''(x) = \qquad f''(0) = \qquad p''(0) = 2c_2$$
$$f'''(x) = \qquad f'''(0) = \qquad p'''(0) = 6c_3$$
$$f^{(iv)}(x) = \qquad f^{(iv)}(0) = \qquad p^{(iv)}(0) = 24c_4$$
$$f^{(v)}(x) = \qquad f^{(v)}(0) = \qquad p^{(v)}(0) = 120c_5$$

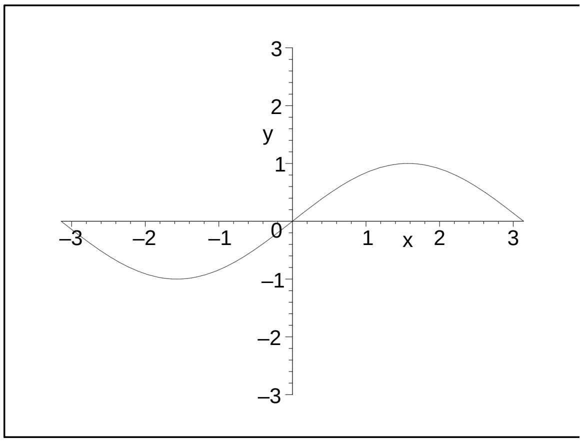
(a). Using the above information, find the values of the coefficients c_0, c_1, c_2, c_3, c_4, and c_5 so that that $f(x)$ and $p(x)$ match up to their 5th derivatives.

(b). Write down the resulting polynomial up to x^5 and simplify.
i.e Write down $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5$ since we're claiming $f(x) \approx p(x)$.

$\sin x \approx$ $\longleftarrow$ fill in polynomial

(c). Graph $f(x) = \sin x$ and the polynomial $p(x)$ from part (b) on your Calculator.
Use the viewing window $-\pi \le x \le \pi$ and $-3 \le y \le 3$.

Sketch the polynomial onto the sine curve below.



How good does the polynomial approximate the sine function near $x = 0$?