$\frac{d}{dx}[c] = 0$ $\qquad$ $\frac{d}{dx}[cx] = c$ $\qquad$ $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$

$\frac{d}{dx}[x] = 1$ $\qquad$ $\frac{d}{dx}[x^n] = nx^{n-1}$ $\qquad$ $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

ALL RULES MUST COME FROM THE LIMIT DEFINITION OF THE DERIVATIVE!

Fill in the spaces/blanks below to prove each of the rules.

1. $f(x) = c$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0 \implies \boxed{\frac{d}{dx}[c] = 0}$

2. $f(x) = x$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1 \implies \boxed{\frac{d}{dx}[x] = 1}$

3. $f(x) = cx$

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{c(x+h) - cx}{h} = \lim_{h\to 0} \frac{cx + ch - cx}{h} = \lim_{h\to 0} \frac{ch}{h} = \lim_{h\to 0} c = c \implies \boxed{\frac{d}{dx}[cx] = c}$

4. $f(x) = x^n$ [See back page for proof]

5. $F(x) = cf(x)$

Then $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$

$= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = c \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = c \cdot \underline{f'(x)} \implies \boxed{\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]}$

6. $F(x) = f(x) + g(x)$ $\qquad$ [Similar proof for $f(x) - g(x)$]

Then $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$

$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x) \implies \boxed{\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]}$

4. $f(x) = x^n$, for n a positive integer.

Multiply the following polynomials to verify a property that we will use later:

$$(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$$

$$= x \cdot (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$$
$$-a \cdot (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + x^{n-4}a^3 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$$

$$= x \cdot x^{n-1} + x \cdot x^{n-2}a + x \cdot x^{n-3}a^2 + x \cdot x^{n-4}a^3 + \ldots + x \cdot x^2a^{n-3} + x \cdot xa^{n-2} + x \cdot a^{n-1})$$
$$-(x^{n-1} \cdot a + x^{n-2}a \cdot a + x^{n-3}a^2 \cdot a + x^{n-4}a^3 \cdot a + \ldots + x^2a^{n-3} \cdot a + xa^{n-2} \cdot a + a^{n-1} \cdot a)$$

$$= x^n + x^{n-1}a + x^{n-2}a^2 + x^{n-3}a^3 + \ldots + x^3a^{n-3} + x^2a^{n-2} + xa^{n-1}$$
$$- \left(x^{n-1}a + x^{n-2}a^2 + x^{n-3}a^3 + x^{n-4}a^4 + \ldots + x^2a^{n-2} + xa^{n-1} + a^n\right)$$

$$= x^n - a^n$$

In other words, $x^n - a^n$ factors as $(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1})$

$f(x) = x^n$, for n a positive integer. Find the derivative at $x = a$.

Then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1})}{x - a}$$

$= \lim_{x \to a} x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1}$ Note: n terms

$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \ldots + a^2a^{n-3} + aa^{n-2} + a^{n-1}$ Note: n terms

$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots + a^{n-1} + a^{n-1} + a^{n-1}$ Note: n terms

$= \underline{\quad n \quad} \cdot a^{n-1}$ Hint: How many a^{n-1}'s do you have?

Replace a with x and we get $f'(x) = n \cdot x^{n-1}$ $\Longrightarrow$ $\boxed{\frac{d}{dx}[x^n] = nx^{n-1}}$

5. $F(x) = f(x) \cdot g(x)$

Then

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) \quad - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) \quad - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) \quad - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x)] + [f(x+h) \cdot g(x) - f(x) \cdot g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot [\quad g(x+h) - g(x) \quad] + g(x) \cdot [\quad f(x+h) - f(x) \quad]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot [g(x+h) - g(x)]}{h} + \frac{g(x) \cdot [f(x+h) - f(x)]}{h}$$

$$= \lim_{h \to 0} f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \quad g(x) \cdot \frac{f(x+h) - f(x)}{h} \qquad [\mathit{Finish}]$$

[See book for proof of the Quotient Rule.]