

Recall the TANGENT LINE PROBLEM: Given a curve $y = f(x)$ and a point $P(a, f(a))$ on the curve, find the equation for the tangent line to the curve $y = f(x)$ at the point P .

[Sketch]

In order to write the equation of a line, we need

(1). Point: $\checkmark$ $P(a, f(a))$ is given

(2). Slope: Not given. Also not given 2 points on the tangent line (we only have 1 point), so we can't compute the slope.

$\Rightarrow$ Approximate the slope of the tangent line by finding the slope of the secant line through P and another arbitrary point Q on the curve.

If x is the x -coordinate of point Q , then the y -coordinate is $f(x)$. ie. $Q(x, f(x))$

So the slope of the secant line through $P(a, f(a))$ and $Q(x, f(x))$ is given by

$$m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x) - f(a)}{x - a}$$

The approximated slope m_{PQ} will get closer to the slope m of the tangent line as:

Q gets closer to P (i.e. $Q \rightarrow P$)

$\Rightarrow x$ gets closer to a (i.e. $x \rightarrow a$)

i.e. $\frac{f(x) - f(a)}{x - a} = m_{PQ} \rightarrow m$ as $x \rightarrow a$ This is a LIMIT!

(In fact, this limit is how we define the tangent line)

Def. The TANGENT LINE to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Ex Given $f(x) = x^2$ at $x = 2$

- (a). Approximate the slope of the tangent line at $x = 2$ by making a table of values for the slope of the secant line m_{PQ} near $x = 2$.

$$P: \text{ At } x = 2, \quad y = f(2) = 2^2 = 4 \quad \Rightarrow \quad P(2, 4)$$

$$Q: \text{ Any point on } y = x^2 \quad \Rightarrow \quad Q(x, x^2)$$

$$\text{So } m_{PQ} = \frac{f(x) - f(2)}{x - 2} = \frac{x^2 - 4}{x - 2}$$

x	1.5	1.75	1.9	1.99	2	2.01	2.1	2.5	2.75
$m_{PQ} = \frac{x^2 - 4}{x - 2}$	3.50	3.75	3.90	3.99	??	4.01	4.1	4.25	4.5

From the table of values, guess the slope of the tangent line to be $m = \underline{4}$.

$$\text{i.e. } \lim_{x \rightarrow 2} m_{PQ} = m = \underline{4}$$

- (b). Verify your guess in part (a) by calculating the limit analytically.

$$m = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} x + 2 = 4$$

- (c). Find the equation of the tangent line to $f(x) = x^2$ at $x = 2$

$$y - 4 = 4(x - 2)$$

- (d). Sketch the tangent line from part (c) on the graph of $f(x) = x^2$ below. Does this line appear to be the tangent line (just barely touching) at P ?

