Ex From last time, we found that $\boxed{\oint_C \frac{z^2}{z^4+4}\,dz = \frac{\pi}{2}}$ using residues,

where C is the closed semicircular path shown:

Ex Find the Cauchy Principal Value of $\int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\,dx$ i.e. find P.V. $\int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\,dx$

Consider $\oint_C \frac{z^2}{z^4+4}\,dz$ where $C = L_R + C_R$ $(R > \sqrt{2})$.

Then $\oint_C \frac{z^2}{z^4+4}\,dz = \frac{\pi}{2}$ from above.

But $\oint_C \frac{z^2}{z^4+4}\,dz = \int_{L_R} \frac{z^2}{z^4+4}\,dz + \int_{C_R} \frac{z^2}{z^4+4}\,dz$

On L_R:

Parameterize: $\int_{L_R} \frac{z^2}{z^4+4}\,dz = \int_{-R}^{R} \frac{x^2}{x^4+4}\,dx$

Take the limit as $R \to \infty$:

On C_R:

Use the $ML-$Theorem to find a bound on $\left|\int_{C_R} \frac{z^2}{z^4+4}\,dz\right|$

$L =$ πR

$$\left|\frac{z^2}{z^4+4}\right| = \frac{|z^2|}{|z^4+4|} \leq \frac{R^2}{R^4-4} = M_R$$ since

Then $0 \leq \left|\int_{C_R} \frac{z^2}{z^4+4}\,dz\right| \leq M_R \cdot L = \frac{R^2}{R^4-4} \cdot \pi R = \frac{R^3\pi}{R^4-4}$

Take the limit as $R \to \infty$

So

$$\begin{aligned}
\frac{\pi}{2} &= \oint_C \frac{z^2}{z^4+4}\,dz \\
&= \int_{L_R} \frac{z^2}{z^4+4}\,dz + \int_{C_R} \frac{z^2}{z^4+4}\,dz \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^4+4}\,dx + \lim_{R\to\infty} \int_{C_R} \frac{z^2}{z^4+4}\,dz \\
&= \text{P.V.} \int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\,dx
\end{aligned}$$

Since $\frac{x^2}{x^4+4}$ is even, the real-valued improper integrals are given by :

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\,dx = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{x^2}{x^4+4}\,dx = \frac{\pi}{4}$$