Some properties and formulas we'll need to prove Taylor's Theorem:

- Formula for $\dfrac{1}{s-z}$ (derived on previous worksheet):

$$\text{i.e.} \quad \boxed{\frac{1}{s-z} = \left[\sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}}\right] + \frac{z^N}{s^N(s-z)} \qquad (*)}$$

- (THEOREM Section 4.43) If f is piecewise continuous on a contour C and $|f(z)| \leq M$ for all z on C then

$$\left|\int_C f(z)\, dz\right| \leq ML$$

  where L is the length of the contour C.

- THEOREM (given without proof): If f is analytic and nonconstant on a closed region R that is bounded by the simple closed contour C, then $|f(z)|$ is guaranteed to have a maximum which will always occur somewhere on the curve C.

TAYLOR'S THEOREM

Suppose f is analytic in the open disk $|z - z_0| < R$. Then in that disk, $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \frac{f'''(z_0)}{3!}(z-z_0)^3 + \ldots$$

i.e. The series converges to $f(z)$ in this disk $|z - z_0| < R$. [Note: R is called the Radius of Convergence .]

PROOF (for $z_0 = 0$)

Let $f(z)$ be analytic in the open disk $|z - z_0| < R$.

Let z be an arbitrary point inside this disk such that $|z| = r$

Define C_0 to be the positively oriented circle $|z| = r_0$ where $r < r_0 < R$.

[Sketch a picture.]

Since f is analytic inside and on C_0, the Cauchy-Integral formula gives:
[Hint: The singularity in the integrand is at z not z_0 and use s as the variable of integration.]

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{s-z}\,ds \\
&= \frac{1}{2\pi i}\int_{C_0} f(s)\frac{1}{s-z}\,ds \\
&= \frac{1}{2\pi i}\int_{C_0} f(s)\left[\ \left[\sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}}\right] + \frac{z^N}{s^N(s-z)}\ \right] ds \qquad \text{by } (*) \\
&= \frac{1}{2\pi i}\int_{C_0}\left[\left[f(s)\sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}}\right] + f(s)\frac{z^N}{s^N(s-z)}\right] ds \\
&= \frac{1}{2\pi i}\int_{C_0}\left[\sum_{n=0}^{N-1} f(s)\frac{z^n}{s^{n+1}}\right] ds + \frac{1}{2\pi i}\int_{C_0} f(s)\frac{z^N}{s^N(s-z)}\,ds \\
&= \sum_{n=0}^{N-1}\left[\frac{1}{2\pi i}\int_{C_0} f(s)\frac{z^n}{s^{n+1}}\,ds\right] + \frac{1}{2\pi i}\int_{C_0} f(s)\frac{z^N}{s^N(s-z)}\,ds \\
&= \sum_{n=0}^{N-1}\left[\ z^n\ \frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{s^{n+1}}\,ds\right] + \frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{(s-z)}\cdot\frac{z^N}{s^N}\,ds \\
&= \sum_{n=0}^{N-1}\left[z^n\ \frac{f^n(0)}{n!}\ \right] + \frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{(s-z)}\cdot\frac{z^N}{s^N}\,ds \qquad \text{by the Cauchy-Integral formula} \\
&= \sum_{n=0}^{N-1}\left[\frac{f^n(0)}{n!}z^n\right] + R_N(z)
\end{aligned}$$

In the limit as $N \to \infty$, the first term becomes the Taylor Series (about $z_0 = 0$).

So all we need to show is that $R_N = \displaystyle \frac{1}{2\pi i}\int_{C_0}\frac{f(s)}{(s-z)}\cdot\frac{z^N}{s^N}\,ds \to 0$ as $N \to \infty$

Recall the picture for **(i).** $f(z)$ analytic in the open disk $|z - z_0| < R$ **(ii).** z an arbitrary point inside this disk such that $|z| = r$ **(iii).** C_0 the positively oriented circle $|z| = r_0$ where $r < r_0 < R$

- $|z|^n = \underline{\; r^n \;}$
- Since s is the variable of integration and we are integrating along C_0, s can be parameterized as $s = \underline{\; r_0 e^{i\theta} \;}$ and therefore $|s| = \underline{\; r_0 \;}$.
- By the theorem given on the first worksheet, since f is analytic and nonconstant on the region bounded by C_0, then $|f(s)|$ will have a maximum M on C_0, i.e. $|f(s)| \leq M$.
- By the triangle inequality, $|s - z| \geq ||s| - |z|| = |\underline{\; r_0 - r \;}| = r_0 - r$. Therefore, $\dfrac{1}{|s - z|} \leq \dfrac{1}{r_0 - r}$

Combining all of the above, we have $\left| \dfrac{1}{2\pi i} \dfrac{f(s)}{(s - z)} \cdot \dfrac{z^N}{s^N} \right| \leq \dfrac{1}{2\pi} \cdot \dfrac{M}{r_0 - r} \cdot \dfrac{r^N}{r_0^N}$

Since the length L of C_0 is $\underline{\; 2\pi r_0 \;}$, by the $ML-$inequality, we have

$$|R_N| = \left| \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{(s - z)} \cdot \frac{z^N}{s^N}\, ds \right| \leq \underline{\frac{1}{2\pi} \frac{M}{r_0 - r} \cdot \frac{r^N}{r_0^N} \cdot 2\pi r_0} = \frac{M r_0}{r_0 - r} \left(\frac{r}{r_0} \right)^N$$

i.e. $$0 \leq |R_N| \leq \frac{M r_0}{r_0 - r} \left(\frac{r}{r_0} \right)^N$$

$$\Rightarrow 0 \leq \lim_{N \to \infty} |R_N| \leq \lim_{N \to \infty} \frac{M r_0}{r_0 - r} \left(\frac{r}{r_0} \right)^N$$

Since $\dfrac{r}{r_0} \underline{\; < 1 \;} \Longrightarrow \displaystyle\lim_{N \to \infty} \left(\frac{r}{r_0} \right)^N = \underline{\; 0 \;}$.

$$\Rightarrow 0 \leq \lim_{N \to \infty} |R_N| \leq \underline{\; 0 \;}$$

Therefore, by $\underline{\text{ the squeeze theorem }}$, $\displaystyle\lim_{N \to \infty} |R_N| = \underline{\; 0 \;}$

Therefore, by taking the limit as $N \to \infty$

$$f(z) = \lim_{N \to \infty} \sum_{n=0}^{N-1} \left[\frac{f^n(0)}{n!} z^n \right] + \lim_{N \to \infty} R_N(z),$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$$

i.e. The Taylor Series does, in fact, converge to $\underline{\; f(z) \;}$ as $N \to \infty$.