1. Complete the steps below to derive a similar formula for the derivative $f'(z_0)$.

(a). Recall, the Cauchy Integral Formula: $\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz}$

(b). Using the Cauchy Integral Formula $\Longrightarrow f(z_0 + \Delta z) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - (\ z_0 + \Delta z\ )}\,dz$

(c). Write down the limit definition of the derivative of f and z_0.

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[f(z_0 + \Delta z) - f(z_0)\right] \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - (z_0 + \Delta z)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz\right] && \text{from (b) and (a)} \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z}\oint_C \left[\frac{f(z)}{z - z_0 - \Delta z} - \frac{f(z)}{z - z_0}\right]dz \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z}\oint_C \left[\frac{f(z)(\ z - z_0\ ) - f(z)(\ z - z_0 - \Delta z\ )}{(z - z_0 - \Delta z)(z - z_0)}\right]dz && \text{From cross-multiplying} \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z}\oint_C \left[\frac{zf(z) - z_0 f(z) - zf(z) + z_0 f(z) + \Delta z f(z)}{(z - z_0 - \Delta z)(z - z_0)}\right]dz \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i \Delta z}\oint_C \frac{\Delta z f(z)}{(z - z_0 - \Delta z)(z - z_0)}\,dz \\
&= \lim_{\Delta z \to 0} \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)}\,dz
\end{aligned}
$$

[Technically we need more formality involving continuity, bounds, and the limit... but just let $\Delta z \to 0$]

$$
= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)(z - z_0)}\,dz
$$

i.e. $\boxed{f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,dz}$

<u>THEOREM Cauchy Integral Formula Extension</u>

Suppose f is analytic in a simply connected domain enclosed by the simple, closed contour C, with positive orientation. Then for any point z_0 interior to C

$$
\boxed{f^n(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz}
$$

[Note: Problem 1 was essentially the proof for $n = 1$.]

$$\boxed{f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz} \qquad \Longleftrightarrow \qquad \boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i \cdot f^n(z_0)}{n!}}$$

This extension gives us the ability to integrate more functions with higher order singularities in the denominator.

2. $\oint_C \frac{e^{iz}}{z^2}\, dz$ where C is the circle $|z| = 1$. $n =$ 1 $z_0 =$ 0 $f(z) =$ e^{iz}

3. $\oint_C \frac{\cos 2z}{(z-1)^5}\, dz$ where C is the circle $|z| = 3$.

4. $\oint_C \frac{z+1}{z^4 + 2iz^3}\, dz$ where C is the circle $|z| = 1$.