CAUCHY-GOURSAT THEOREM [Given previously.]

If a function f is analytic at all points interior to and on a simple closed contour C, then $\oint_C f(z)\,dz = 0$.

THEOREM CAUCHY-GOURSAT EXTENSION 1 [Used for]

If a function f is analytic throughout a simply connected domain D, then $\oint_C f(z)\,dz = 0$ for every closed contour C lying in D.

What is the difference? [Sketch picture.]

Informal Proof:

EX: Evaluate the following integral where C is the contour given in the following sketch.

$$\int_C \tan z\,dz$$

Let D be the given domain with one hole (doubly-connected).

Let C be the outer boundary of D (in the ccw direction).

Let C_1 be the inner boundary of D (in the cw direction).

Then the entire boundary ∂D is comprised of ______________
and note that as you traverse C and C_1 in the given orientations,
that the domain D is always to the _________ of the contour.

Introduce a line L_1 that connects C to C_1.
Allow the the path along L_1 and L_2 to travel in both directions.

If you now traverse the contour $\Gamma = C + L_1 + (C_1) - L_1$, what type of domain does it enclose?

Evaluate $\displaystyle\int_\Gamma f(z)\,dz$

This idea can be extended for multiply connected domains (more than one hole)

<u>Theorem</u> <u>Cauchy-Goursat Extension 2</u> [Used for]
Suppose that

(a). C is a simple closed contour oriented in the *counterclockwise* direction.

(b). C_k $(k = 1, 2, \ldots, n)$ are simple closed contours interior to C, oriented in the *clockwise* direction, that are disjoint with no common interior points.

If a function f is analytic on all of these contours and throughout the multiply-connected domain D consisting of the points inside C and exterior to each C_k, then

[Sketch picture.]

Ex: Given the following picture,

(a). Use the last theorem to evaluate: $\int_{C_2} f(z)\,dz + \int_{-C_1} f(z)\,dz =$

(b). Rewrite the last result to involve contour integrals of C_1 and C_2 [instead of $-C_1$].

Based on the last problem, fill in the blanks (except for the name) to the following corollary.

Corollary [____]
Let C_1 and C_2 denote simple closed curves oriented in the ____________________ direction, where C_1 is interior to C_2. If a function f is ____________ in the closed region consisting of those contours and all the points between them, then

In other words,

From Part I, problem 2, we saw that $\int_C \frac{1}{z}\,dz =$ _________ for both the unit circle and the square.

This last corollary proves that $\int_C \frac{1}{z}\,dz =$ _________ for any positively oriented curve about the origin.

Ex: From 8th Edition Section 4.2 Example 2 & Exercise 10b or 9th Edition Section 4.46 Exercise 13, we have the following result:

$$\oint_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 0, & n = \pm 1, \pm 2, \ldots \\ 2\pi i, & n = 1 \end{cases}$$

Where C is the circle of radius R centered at z_0. i.e. $C : z = z_0 + Re^{i\theta}$.

Use deformation of path to fill in the blank of the more general result:

$$\oint_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 0, & n = \pm 1, \pm 2, \ldots \\ 2\pi i, & n = 1 \end{cases}$$

Where C is __________ positively oriented simply closed contour surrounding __________ .

1. Use the above result to evaluate the following integral where C is the triangle with vertices at $i, 2-i$, and $-2-i$.

$$\oint_C \frac{3}{2z-1}\,dz$$

2. Verify that your answer is correct by evaluating the same integral using parameterization. Rather than parameterize the triangle, deformation of path allows you to choose a "nicer" contour C_1. What would be a good choice?

$$\oint_C \frac{3}{2z-1}\,dz = \oint_{C_1} \frac{3}{2z-1}\,dz$$

Does your answer match up with #1? If not, go back and see if you can find your mistake (it might be in #1, rather than in #2).

3. Given the following integral, first use partial fraction decomposition to rewrite the integrand. Then use any known theorems to evaluate the integrals without using parameterization. Let C_1 be the same wisely chosen contour from problem 2.

$$\oint_{C_1} \frac{4z-9}{(2z-1)(z-4)}\,dz$$