1. Fill in the blanks to derive an expression for the inverse sine function (i.e. $\sin^{-1} z$).

Let $w = \sin^{-1} z$, then $z = \sin w$. [We want to solve for w since it equals $\sin^{-1} z$.]

Then $z = \underline{\dfrac{e^{iw} - e^{-iw}}{2i}}$ by the definition of $\sin w$.

Multiply both sides by $2ie^{iw}$:

$$\begin{aligned} 2ize^{iw} &= \frac{e^{iw} - e^{-iw}}{2i} \cdot 2ie^{iw} \\ &= \underline{e^{2iw} - 1} \end{aligned}$$

$\Rightarrow e^{2iw} - 2ize^{iw} - 1 = 0$ But $e^{2iw} = (\underline{e^{iw}})^2$

So $\left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 = 0$ is a quadratic equation in $\underline{e^{iw}}$.

[Given a complex quadratic equation $az^2+bz+c = 0$, the quadratic formula (for complex variables) is $\underline{z = \dfrac{-b + (b^2 - 4ac)^{1/2}}{2a}}$ See Exercise 8(a), Sec. 10. Be sure to use proper notation.]

Applying this formula to the above equation yields:

$$e^{iw} = \frac{2iz + \left[(-2iz)^2 - 4(1)(-1)\right]^{1/2}}{2(1)} = \frac{2iz + \left[-4z^2 + 4\right]^{1/2}}{2} = \frac{2iz + \left[4(1 - z^2)\right]^{1/2}}{2} = \frac{2iz + 2\left(1 - z^2\right)^{1/2}}{2}$$

So $e^{iw} = iz + \underline{\left(1 - z^2\right)^{1/2}}$ [Note: The RHS is double-valued since there are exactly 2 roots.]

Solve for w by taking the logarithm of both sides:

$$\log(e^{iw}) = \log\left[iz + \left(1 - z^2\right)^{1/2}\right]$$

$$\underline{iw} = \log\left[iz + \left(1 - z^2\right)^{1/2}\right]$$

$$w = \underline{-i\log\left[iz + \left(1 - z^2\right)^{1/2}\right]}$$

Therefore, since $w = \underline{\sin^{-1} z}$ from the very beginning, we have $\boxed{\sin^{-1} z = -i\log\left[iz + \left(1 - z^2\right)^{1/2}\right]}$

Similarly, $\boxed{\cos^{-1} z = -i\log\left[z + i\left(1 - z^2\right)^{1/2}\right]}$ $\boxed{\tan^{-1} z = \dfrac{i}{2}\log\dfrac{i + z}{i - z}}$

2. Consider the expression for $\sin^{-1} z$ above. Is it single-valued, double-valued, or multiple(more than 2)-valued? [You may need to check your answer after doing problem #3.]

It is multiple-valued because of the logarithm.

3. Fill in the blanks to find $\sin^{-1}\sqrt{5}$.

$$\begin{aligned}
\sin^{-1}\sqrt{5} &= -i\log\left[i\underline{\quad\sqrt{5}\quad} + (1 - \underline{\quad 5 \quad})^{1/2}\right] \\
&= -i\log\left[i\sqrt{5} + (-4)^{1/2}\right] \qquad \text{Luckily, we know } (-4)^{1/2} = \underline{\quad \pm 2i \quad} \text{ w/o needing polar representation.} \\
&= -i\log\left[i\sqrt{5} \pm \underline{\quad 2i \quad}\right] \\
&= -i\log\left[(\sqrt{5}\pm 2)i\right] \\
&= -i\left[\ln|\underline{\quad(\sqrt{5}\pm 2)i\quad}| + i\arg(\underline{\quad(\sqrt{5}\pm 2)i\quad})\right] \\
&= -i\left[\ln|(\sqrt{5}\pm 2)| + i\arg((\sqrt{5}\pm 2)i)\right] \qquad \text{since } |i| = 1.
\end{aligned}$$

But $(\sqrt{5}\pm 2)i$ are both purely $\underline{\quad\text{imaginary}\quad}$ AND $\sqrt{5}\pm 2$ are both positive.

So they both have the same argument:

$$\begin{aligned}
&= -i\left[\ln(\sqrt{5}\pm 2) + i\left(\underline{\quad\frac{\pi}{2} + 2n\pi\quad}\right)\right], \quad n = 0, \pm 1, \pm 2, \ldots \\
&= \left(\frac{\pi}{2} + 2n\pi\right) - i\ln(\sqrt{5}\pm 2), \quad n = 0, \pm 1, \pm 2, \ldots
\end{aligned}$$

We could stop here, but to see how the book writes some answers, complete the following extra steps.

$$\sqrt{5} - 2 = (\sqrt{5} - 2)\cdot\frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{5-4}{\sqrt{5}+2} = \frac{1}{\sqrt{5}+2} \qquad \text{[Simplify.]}$$

So

$$\ln(\sqrt{5}-2) = \ln\left(\frac{1}{\sqrt{5}+2}\right) = \ln(\sqrt{5}+2)^{-1} = \underline{\quad -\ln(\sqrt{5}+2) \quad}$$

In other words, $\ln(\sqrt{5}\pm 2) = \pm\underline{\quad\ln(\sqrt{5}+2)\quad}$.

So the answer above can be written as $\sin^{-1}(\sqrt{5}) = \left(\frac{\pi}{2} + 2n\pi\right) \pm i\ln(\sqrt{5}+2), \quad n = 0, \pm 1, \pm 2, \ldots$

Sketch the values in the complex plane.

[Go back to question number 2 – did you answer it correctly?]

4. Look in your book and write down the definitions for the inverse hyperbolic functions in terms of logarithms.
8th edition, Section 3.36 or 9th edition, Section 3.40

$\sinh^{-1} z = \quad \log\left[z + \left(z^2+1\right)^{1/2}\right]$	$\cosh^{-1} z = \quad \log\left[z + \left(z^2-1\right)^{1/2}\right]$	$\tanh^{-1} z = \quad \dfrac{1}{2}\log\dfrac{1+z}{1-z}$

5. Find all the values of $\cosh^{-1}(3i)$. [Be careful when finding the argument(s).]

Sketch the values in the complex plane.

6. If we make a branch cut so that the inverse trigonometric functions are single-valued and analytic, then the derivatives are defined as: [Look up any missing formulas.]

$$\frac{d}{dz}\left[\sin^{-1} z\right] = \frac{1}{(1-z^2)^{1/2}} \qquad \frac{d}{dz}\left[\cos^{-1} z\right] = -\frac{1}{(1-z^2)^{1/2}} \qquad \frac{d}{dz}\left[\tan^{-1} z\right] = \frac{1}{1+z^2}$$

Fill in the blanks to derive the derivative of $\sin^{-1} z$

Let $w = \sin^{-1} z$. [So $\frac{dw}{dz} = \frac{d}{dz}\left[\sin^{-1} z\right]$ and therefore we need to find $\frac{dw}{dz}$]

Then $\underline{\quad \sin w \quad} = z$

Implicitly differentiate both sides [Remember w is a function of z.]

$$\frac{d}{dz}\left[\sin w\right] = \frac{d}{dz}\left[z\right]$$

$$\underline{\quad \cos w \frac{dw}{dz} \quad} = 1$$

Then $\frac{dw}{dz} = \frac{1}{\cos w}$ **(*)** [But we need $\frac{dw}{dz}$ as a function of z.]

From the identity $\cos^2 w + \sin^2 w = 1$, we get $\cos w = \underline{\quad (1-\sin^2 w)^{1/2} \quad}$.

But from above, $\sin w = z$, so $\cos w = (1-\sin^2 w)^{1/2} = \underline{\quad (1-z^2)^{1/2} \quad}$

Substitute this expression into **(*)** to get $\frac{dw}{dz} = \frac{1}{(1-z^2)^{1/2}}$. i.e. $\boxed{\frac{d}{dz}\left[\sin^{-1} z\right] = \frac{1}{(1-z^2)^{1/2}}}$

Homework 8th Edition Section 3.36[9th Edition, Section 3.40], p. 114: #1(sketch values in the complex plane), 2, 3, 5

Additional Problem: I. Given $\cos^{-1}\left(\frac{5}{3}\right)$,

(a). Find all values.

(b). Find the value using the principal branch of $\log z$ and the principal branch of $z^{1/2}$ or state why it is undefined.

(c). Find the value using the principal branch of $\log z$ and the branch $z^{1/2} = \sqrt{r}e^{i\theta/2}$ $(r > 0, 0 < \theta < 2\pi)$ for $z^{1/2}$ or state why it is undefined.

(d). Find the derivative of $\cos^{-1}(z)$ at $z = \frac{5}{3}$ using the branches from part (a) or state why it is undefined.

(e). Find the derivative of $\cos^{-1}(z)$ at $z = \frac{5}{3}$ using the branches from part (b) or state why it is undefined.