Recall $e^{\log z} = z$ for all $z \in \mathbb{C}$, $z \neq 0$, but $\log e^z = z +$ ___________

So $f(z) = e^z$ and $g(z) = \log z$ do not satisfy the cancelation properties of _____________ functions. This happens because $f(z)$ is not one-to-one on its domain $\mathbb{C}$.

However, if we restrict the domain of $f(z) = e^z$ to the fundamental region $-\infty < x < \infty, -\pi < y \leq \pi$, then $f(z)$ is ________________ on its domain and an inverse function exists, which is $f^{-1} =$ ____________.

Properties

(a). $\log z_1 z_2 =$ ___________________ **(b).** $\log\left(\dfrac{z_1}{z_2}\right) =$ ___________________

<u>Proof</u> (a) $\log z_1 + \log z_2 = \ln|z_1| + i \arg z_1 +$ ____________________

$= \ln|z_1| + \ln|z_2| + i($____________________$)$

$=$ ______________ $+\, i \arg z_1 z_2$ by properties of ln, abs. val. (for real #'s) and arg (Sec. 1.8).

$= \log($___________$)$ ■

1. Verify the above properties for $z_1 = -1 + i$ and $z_2 = -i$.

(a). LHS =

RHS =

(b). LHS =

RHS =

2. Find two complex numbers to show that the above properties do not necessarily hold for the principle value Log z.

(a). LHS =

RHS =

(b). Note: 1(b) should already show this.

More Properties

(a). $z^n = e^{n \log z}$, for $n = 0, \pm 1, \pm 2, \ldots$

(b). $z^{\frac{1}{n}} = e^{\frac{1}{n} \log z}$, for $z \neq 0$, $n = 0, \pm 1, \pm 2, \ldots$ [Multiple-valued]

PROOF (a) Let $z = re^{i\theta}$, then

$$\begin{aligned} e^{n \log z} &= e^{n[\qquad\qquad\qquad]} \\ &= e^{n \ln r} \cdot e^{in\theta} \cdot e^{i2n\pi} \\ &= e^{\qquad} \cdot e^{in\theta} \cdot \underline{\qquad\qquad} \\ &= r^n e^{in\theta} \\ &= \underline{\qquad\qquad} \quad \blacksquare \end{aligned}$$

PROOF (b) Case 1: $n \in \mathbb{Z}^+$

Let $z = re^{i\theta}$, then

$$\begin{aligned} \exp\left(\tfrac{1}{n} \log z\right) &= \exp\left(\tfrac{1}{n}[\ln r + i(\Theta + 2k\pi)]\right) && k = 0, \pm 1, \pm 2, \ldots \\ &= \exp\left(\frac{1}{n} \ln r + i\left(\qquad\qquad\right)\right) && k = 0, \pm 1, \pm 2, \ldots \\ &= \underline{\qquad\qquad\qquad} \cdot e^{\left(i\left(\frac{\Theta + 2k\pi}{n}\right)\right)} && k = 0, \pm 1, \pm 2, \ldots \\ &= e^{\left(\ln r^{1/n}\right)} \cdot e^{\left(i\left(\frac{\Theta + 2k\pi}{n}\right)\right)} && k = 0, \pm 1, \pm 2, \ldots \\ &= \underline{\qquad\qquad} e^{\left(i\left(\frac{\Theta}{n} + \frac{2k\pi}{n}\right)\right)} && k = 0, \pm 1, \pm 2, \ldots \\ &= z^{1/n} \text{ by definition and since} \end{aligned}$$

it yields distinct values only for $k = 0, 1, 2, \ldots,$ ____________ $\blacksquare$

Case 2: $n \in \mathbb{Z}^-$ [One of the assigned homework problems]

Since the two properties above hold for powers that are integers and their reciprocals, we suspect that it might hold for _________ complex numbers c.

DEF The Complex Power Function ($z \neq 0$) is defined as $z^c =$ ____________ for any $c \in \mathbb{C}$.

Is this function single-valued or multiple-valued?

3. Evaluate the following and sketch the answer(s) in the complex plane:

(a). i^{3i}

$$\begin{aligned} i^{3i} &= e^{3i\log i} = e^{3i}[\qquad\qquad\qquad\qquad] \\ &= e^{3i}[\ln 1 + i(\pi/2 + 2n\pi)] \\ &= e^{3i}[\qquad\qquad\qquad\qquad] \\ &= e^{-3\pi/2 - 6n\pi} \end{aligned}$$

(b). $(-\sqrt{3} - i)^i$

4. Use properties of the exponential function to show that $z^{-c} = \dfrac{1}{z^c}$

5. Evaluate $(-\sqrt{3} - i)^{-i}$

<u>DEF</u>	P.V. $z^c = e^{c\text{Log}\, z}$, is the Principle Value of z^c

Using the Principle Branch of Log z [i.e. $|z| > 0$ and $-\pi$________Arg z________π],
P.V. z^c defines the Principle Branch of z^c.

6. Find P.V. $(-\sqrt{3} - i)^i$

Recall, that for <u>*any*</u> branch of $\log z$ $(r > 0, \alpha < \theta <$ ____________$)$,

$\log z = \ln r + i\theta$ is ____________ -valued.

7. Using such a branch and the definition of z^c, fill in the steps to show that $\frac{d}{dz}[z^c] = cz^{c-1}$

$\frac{d}{dz}[z^c] = \frac{d}{dz}\left[e^{c\log z}\right]$		by definition of the power function
$= e^{c\log z} \cdot \frac{d}{dz}[\qquad\qquad]$		by the chain rule
$= e^{c\log z} \cdot c \cdot$		by the derivative of $\log z$
$= ce^{c\log z} \cdot \frac{1}{z}$		
$= ce^{c\log z} \cdot z^{-1}$		by properties of z^c
$= ce^{c\log z} \cdot e$		by definition of the power function
$= ce^{c\log z - \log z}$		by properties of $\log z$
$= ce^{(\qquad\qquad)\log z}$		
$= cz^{c-1}$		by definition of the power function