DEF The **derivative of f at z_0**, denoted $f'(z_0)$ is $$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ [Fill in missing term.]

An alternate form is $$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$ [Fill in missing limit.]

Note: $f'(z)$ as a function can be represented as $f'(z) = \dfrac{df}{dz} = \lim_{\Delta z\to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ [Fill in missing limit.]

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1. Given $f(z) = z^2$, use the limit definition to find $f'(z)$.

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} =$$

2. Given $f(z) = |z|^2$, use the limit definition to [Hint: At some point use $|z|^2 = z\bar{z}$]

(a). Find $f'(0)$ [Use version 1.]

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} =$$

i.e. $f'(0) = $ ___0___ for $f(z) = |z|^2$.

[Continuation of problem 2 with $f(z) = |z|^2$.]

(b). Show that $f'(z)$ does not exist for all $z \neq 0$. [Use version 2.]

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\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\overline{z+\Delta z}) - z\overline{z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\ \overline{z} + \overline{\Delta z}\ ) - z\overline{z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z\overline{z} + z\overline{\Delta z} + \overline{z}\Delta z + \Delta z\overline{\Delta z} - z\overline{z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{z\overline{\Delta z} + \overline{z}\Delta z + \Delta z\overline{\Delta z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left(\frac{z\overline{\Delta z}}{\Delta z} + \overline{z} + \overline{\Delta z} \right) = \lim_{\Delta z \to 0} \left(\overline{\Delta z} + \overline{z} + \frac{z\overline{\Delta z}}{\Delta z} \right) \\
&= \lim_{\Delta z \to 0} \overline{\Delta z} + \lim_{\Delta z \to 0} \overline{z} + \lim_{\Delta z \to 0} \frac{z\overline{\Delta z}}{\Delta z} \\
&= 0 + \overline{z} + z \cdot \lim_{\Delta z \to 0} \left(\frac{\overline{\Delta z}}{\Delta z} \right)
\end{aligned}
$$

But the $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ DNE since if $\Delta z = \Delta x + i\Delta y$ approaches 0 along two (well-chosen) different paths, you get different limits.

[Shown the other day as $\lim_{z \to 0} \frac{\overline{z}}{z}$ DNE.]

Therefore, since this limit DNE, the derivative $f'(z)$ ___DNE___ for $z \neq 0$.

3. Consider problem 1 again where $f(z) = z^2$.

(a). What did you get for $f'(z)$? Would your intuition have suggested this would have been the answer?

(b). Let $z = x + iy$ and write $f(z)$ in the form $u(x, y) + iv(x, y)$.

(c). From part (b) find the following partial derivatives u_x, u_y, v_x, and v_y.

(d). Let $z = x + iy$ and write $f'(z)$ in the form $s(x, y) + it(x, y)$.

(e). Can you see a connection between the partial derivatives found in part (c) and the form of the derivative in part (d)?

4. Consider problem 2 again where $f(z) = |z|^2$.

(a). Let $z = x + iy$ and write $f(z)$ in the form $u(x, y) + iv(x, y)$.

(b). From part (a) find the following partial derivatives u_x, u_y, v_x, and v_y.

(c). What is different about these partial derivatives in part (b) and those found in problem 3 part (c)?

Comments about $f(z) = |z|^2 = x^2 + y^2$ or $f(z) = u(x,y) + iv(x,y)$ where $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

$f'(z) = \frac{df}{dz}$ **DNE** (for $z \neq 0$) even though:

- The partial derivatives of u and v exist: $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$, and $\frac{\partial v}{\partial y} = 0$
- The partial derivatives of f exist: $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$
- The function $f(z) = |z|^2$ is continuous for all z. [i.e. Continuity does not imply differentiability.]

Can we find conditions on $u(x,y)$ and $v(x,y)$ that will guarantee the existence of the derivative $f'(z)$? [Let's try:]

Assume that $f'(z)$ exists and let $f(z) = u(x,y) + iv(x,y)$. Also, let $\Delta z = \Delta x + i\Delta y$. Then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$= \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{u(x+\Delta x, y+\Delta y) - u(x,y) + i\,[\ v(x+\Delta x, y+\Delta y) - v(x,y)\ ]}{\Delta x + i\Delta y}$$

Since we have assumed that the derivative exists, then the limit must also exist and be independent of path.

Path 1: $\Delta y = 0, \Delta x \to 0 \Longrightarrow f'(z)$

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i\,[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad (*)$$

Path 2: $\Delta x = 0, \Delta y \to 0 \Longrightarrow f'(z)$

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i\,[v(x, y+\Delta y) - v(x,y)]}{i\Delta y} \qquad \text{mult. by } \frac{-i}{-i}$$

$$= \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} - i \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$= \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} \qquad (**)$$

Since these limits must be equal for the derivative to exist $\Rightarrow$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann Equations