Ex: Answer the following questions quickly, without work.

What are the square roots of 4? What are the square roots of -4? What is the cube root of -27?

Ex: What are the square roots of 4? [Now we'll show the work in detail.]

i.e. They are solutions to the equation

$x^2 - 4 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm\sqrt{4} \quad \Rightarrow \quad x = \pm 2$ [All the work is shown for you. No additional work necessary.]

Ex: What are the square roots of -4?

i.e. They are solutions to the equation [Add work similar to the above example to find the roots.]

$x^2 - (-4) = 0 \qquad x^2 + 4 = 0 \qquad \Rightarrow$

Ex: What is the cube root of -27?

i.e. They are solutions to the equation [Add work similar to the above example to find the roots.]

$x^3 + 27 = 0 \qquad \Rightarrow$

Do you think $x =$ ___-3___ is the only cube root of -27?

Ex: What is the fifth root of 32?

i.e. They are solutions to the equation [Add work similar to the above example to find the roots.]

___$x^5 - 32$___ $= 0 \qquad \Rightarrow$

Do you think $x =$ ___2___ is the only fifth root of 32?

Generalize:

What is the n^{th} root of a (complex) number z_0?

i.e. They are solutions to the equation $\underline{\quad z^n - z_0 \quad} = 0$

$$\Rightarrow \quad z^n = z_0$$

$$\Rightarrow \quad z = z_0^{1/n}$$

The n^{th} root(s) of a (complex) number z_0 are ***all*** the values z (unknown) such that

$z = z_0^{1/n}$ or equivalently $z^n = z_0$ where z_0 and n are given.

For complex numbers, find all $\underline{z = re^{i\theta}}$ such that $\underline{z^n = z_0}$

[Note: If z is the unknown, then so are r and $\underline{\quad \theta \quad}$.]

Rewrite the equation $z^n = z_0$ in exponential form: $\Rightarrow \underline{\quad r^n e^{in\theta} \quad} = r_0 e^{i\theta_0}$

For these two complex numbers to be equal, $r^n = \underline{\quad r_0 \quad}$ and $n\theta = \theta_0 + 2k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$.

Solve for the unknowns r and θ: $r = \underline{\quad \sqrt[n]{r_0} = r_0^{1/n} \quad}$ and $\theta = \underline{\quad \frac{\theta_0}{n} \quad} + \frac{2k\pi}{n}$ for $k = 0, \pm 1, \pm 2, \ldots$.

[Note: For complex numbers $r > 0$ (specifically, $r_0 > 0$), so $\sqrt[n]{r_0}$ is a real-valued root.]

Substitute these values back into $z = re^{i\theta}$. [The root(s) we were looking for.]

i.e. The n^{th} root of a complex number is given by

$$\boxed{z_0^{1/n} = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} \text{ for } k = 0, \pm 1, \pm 2, \ldots}$$

Ex: **(a).** Find ***all*** of the cube roots of -27 i.e. Find all z such that $z^3 =$ __-27__.

Write $z_0 = -27$ in exponential form using the principle argument: $z_0 = -27 = 27e^{i(\pi)}$ [Sketch, if helpful.]

So $r_0 =$ __27__ and $\theta_0 =$ __π__. [Use these specific values for r_0 and θ_0 in the formula on p.2]

$$\Rightarrow (-27)^{1/3} = \underline{\sqrt[3]{27}}\, e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)} \text{ for } k = 0, \pm 1, \pm 2, \ldots$$

Use formula on p. 2.

$$= \boxed{3e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)} \text{ for } k = 0, \pm 1, \pm 2, \ldots}$$

Simplify.

(b). List out the roots for $k = 0, 1, 2, 3, 4$

$k = 0 : 3e^{i\pi/3}$

$k = 1 : 3e^{i(\pi/3 + 2\pi/3)} = 3e^{i\pi}$

$k = 2 : 3e^{i5\pi/3}$

$k = 3 : 3e^{i7\pi/3}$

$k = 4 : 3e^{i3\pi}$

(c). Use the space above to lot these roots in the complex plane.

Observations: Specific and Generalized

- Roots lie on a circle centered at the __origin__ given by $|z| =$ __3__. **Lie on circle $|z| = \sqrt[n]{r_0}$.**
- Roots are evenly spaced, every __$\frac{2\pi}{3}$__ radians, around the circle. **Spaced every __$\frac{2\pi}{n}$__ radians.**
- Roots form the vertices of a regular (equilateral) triangle. **Form vertices of a regular $n-$gon for $n \geq 3$.**
- When $k =$ __3__ the roots start repeating. **When __$k = n$__ the roots start repeating.**

Only need to consider $k = 0, 1, 2, \ldots,$ __$n-1$__ for n distinct roots.