Basic Properties

	Additive	**Multiplicative**
Commutative:	$z_1 + z_2 = z_2 + z_1$	$z_1 z_2 = z_2 z_1$
Associative:	$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$	$(z_1 z_2) z_3 = z_1(z_2 z_3)$
Distributive:	$z_3(z_1 + z_2) = z_3 z_1 + z_3 z_2$	
Identities:	$0 + z = z$	$1 \cdot z = z$
Inverses:	For all $z \in \mathbb{C}$ $\exists$ a unique additive inverse $-z$ such that $z + (-z) = 0$	For all $z \in \mathbb{C}, z \neq 0$ $\exists$ a unique multiplicative inverse z^{-1} such that $z \cdot z^{-1} = 1$

[Proofs follow from properties of real numbers and the definitions of complex numbers along with addition and multiplication.]

1. If $z = (x, y)$, then clearly (from properties of real numbers) the additive inverse $-z =$ ____________.

2. But if $z = (x, y)$, then z^{-1} is not so clear.
Let's find it by letting $z^{-1} = (a, b)$ and finding a and b such that $z \cdot z^{-1} = 1$:

$\Longrightarrow (x, y)(a, b) = (1, 0)$

$\Longrightarrow$ (____________, ____________) $= (1, 0)$ by the definition of multiplication.

Then $ax - by = 1$ and $bx + ay = 0$ which are 2 (nonlinear) equations and 2 unknowns a and b

From the first equation: $ax = 1 + by \Longrightarrow a =$ ____________

Substitute into the second equation:

$bx +$ ____________ $\cdot y = 0 \quad \Longrightarrow \quad bx^2 + (1 + by) \cdot y = 0 \quad \Longrightarrow \quad bx^2 +$ ____________ $= 0$

$bx^2 + by^2 =$ ________ $\quad \Longrightarrow \quad b($____________$) = -y \quad \Longrightarrow \quad \boxed{b = \dfrac{-y}{x^2 + y^2}}$

Substitute this expression for b back into the expression for a:

$$a = \frac{1 + by}{x} \quad \Longrightarrow \quad a = \frac{1 + \qquad y}{x} \quad \Longrightarrow \quad a = \frac{1 + \frac{-y}{x^2 + y^2} y}{x} \cdot \frac{x^2 + y^2}{x^2 + y^2}$$

$$a = \frac{\qquad}{x(x^2 + y^2)} \quad \Longrightarrow \quad a = \frac{x^2}{x(x^2 + y^2)} \quad \Longrightarrow \quad \boxed{a = \qquad}$$

That is $\boxed{z^{-1} = \left(\dfrac{x}{x^2 + y^2}, \dfrac{-y}{x^2 + y^2} \right) = \dfrac{x}{x^2 + y^2} + i \dfrac{-y}{x^2 + y^2}}$

<u>Def</u>

<u>Subtraction</u> $z_1 - z_2 = z_1 + (-z_2) =$

<u>Division</u> $\dfrac{z_1}{z_2} = z_1 z_2^{-1}, \quad z_2 \neq 0$

By definition, $\dfrac{z_1}{z_2} = (x_1, y_1)\left(\dfrac{x_2}{x_2^2+y_2^2}, \dfrac{-y_2}{x_2^2+y_2^2}\right) = (x_1 + iy_1)\left(\dfrac{x_2}{x_2^2+y_2^2} + i\dfrac{-y_2}{x_2^2+y_2^2}\right)$, then FOIL.

In practice, rationalize the denominator .

<u>Ex</u>: Divide (and simplify): $\dfrac{-2+i}{4-3i}$

(a). By definition

(b). By rationalizing

More Properties

(a). $\dfrac{1}{z} = z^{-1}$

(b). $\dfrac{z_1}{z_2} = z_1 \cdot \dfrac{1}{z_2}$

(c). $\dfrac{1}{z_1 z_2} = (z_1 z_2)^{-1} = z_1^{-1} z_2^{-1} = \dfrac{1}{z_1} \cdot \dfrac{1}{z_2}$

Proof (a): Let $z = x + iy$.

Then $\dfrac{1}{z} = \dfrac{1}{\_\_\_\_\_} = \dfrac{1}{x+iy} \cdot \dfrac{}{\_\_\_\_\_} = \dfrac{x-iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} + i \qquad = z^{-1}$ (by definition of z^{-1}, see p.1) ■

(b) and (c) proved similarly [But don't prove.]

Another Property

If $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$ (or both).

Proof: Assume $z_2 \neq 0$. Then __________ exists.

Let $z_1 z_2 = 0$ and multiply by $z_2^{-1} \Longrightarrow z_1 z_2 \cdot z_2^{-1} = 0 \cdot z_2^{-1} \Longrightarrow z_1 \cdot$ ________ $= 0 \Longrightarrow z_1 = 0$.

Similarly, if $z_1 \neq 0$, then z_2 must equal 0. ■

Graphical Vector Representation

The complex number $z = (x, y) = x + iy$ can be represented in the plane by the vector $\langle x, y \rangle$. [Sketch below]

Def $|z| = \sqrt{x^2 + y^2}$ is the ____________________ of the complex number z. [i.e. __________________]

Even More Properties

(a). $|\bar{z}| = |z|$

(b). $z\bar{z} = |z|^2$

(c). $z + \bar{z} = 2\,\mathrm{Re}(z)$

(d). $z - \bar{z} = 2i\,\mathrm{Im}(z)$

(e). $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(f). $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

(g). $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

(h). $|z_1 z_2| = |z_1|\,|z_2|$

(i). $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(j). $\mathrm{Re}(z) \le |z|$

(k). $\mathrm{Im}(z) \le |z|$

Proof (f): Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $\overline{z_1 \cdot z_2} = \overline{(x_1 + iy_1) \cdot (x_2 + iy_2)} = \overline{\qquad\qquad\qquad + i(x_1y_2 + x_2y_1)} = x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1)$

Also, $\overline{z_1} \cdot \overline{z_2} = (x_1 - iy_1) \cdot (x_2 - iy_2) = x_1x_2 - y_1y_2 - i(\qquad\qquad)$ i.e. RHS = LHS ■

Proof (h): Consider $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2})$ by property (b)

Then $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\overline{z_1}\;\overline{z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) =$ ______________

Take the square root: $|z_1 z_2| = |z_1||z_2|$ ■

Graphical Vector Addition and Subtraction [Sketch]

<u>Ex</u>: Given $z_1 = 4 + 2i$ and $z_2 = 1 + 3i$

(a). Sketch z_1, z_2, and $z_1 + z_2$. Verify your sketch by computing $z_1 + z_2$.

(b). Sketch z_1, z_2, and $z_1 - z_2$. Verify your sketch by computing $z_1 - z_2$.

(c). Find the distance between z_1 and z_2.

<u>Ex</u>: Describe the set of points that satisfy $|z - 1 + 2i| = 3$.