

1. Express the following in the form $a + ib$. $\tanh^{-1}(1 + 2i) = \frac{1}{2} \left[\ln \sqrt{2} + i \left(\frac{3\pi}{4} + 2n\pi \right) \right], n \in \mathbb{Z}$

2. Solve $\sin z = \sqrt{2}$ using inverse trigonometric functions.

3. Evaluate the following integrals

(a). $\int_1^2 \frac{2i}{t} - t \, dt = 2i \ln 2 - \frac{3}{2}$

(b). $\int_C G(x, y) \, dx$, and $\int_C G(x, y) \, dy$, for $G(x, y) = 3x^2 + 6y^2$ and $C : x = t, y = t^2 + 1$ for $-1 \leq t \leq 0$
 $\int_C 3x^2 + 6y^2 \, dx = -\frac{61}{5}$ and $\int_C 3x^2 + 6y^2 \, dy = \frac{31}{2}$

(c). $\int_C x^6 y \, ds$ where $C : x = t, y = \frac{1}{t}$ $1 \leq t \leq 2$ $\frac{17\sqrt{17} - 2\sqrt{2}}{6}$

(d). $\int_C (x^2 + iy^3) \, dz$ where C is the straight line from $z = 1$ to $z = i$. $-\frac{7}{12} + \frac{1}{12}i$

(e). $\oint_C ze^z \, dz$ where C is the square with vertices $z = 0, z = 1, z = 1 + i, z = i$. 0

4. By finding an antiderivative, evaluate the following integrals, where the path is an arbitrary contour between the indicated limits of integration.

(a). $\int_0^{3+i} z^2 \, dz = 6 + \frac{26}{3}i$ (b). $\int_{1-i}^{1+2i} ze^{z^2} \, dz = 1/2 (-1 + e^{-3+6i}) e^{-2i}$ (c). $\int_{\pi}^{\pi+2i} \sin \frac{z}{2} \, dz = 2 \sin(i)$

5. Given the following functions, determine for which ones The Cauchy-Goursat Theorem applies so that $\oint_C f(z) \, dz = 0$. For those which it does not apply, evaluate the integral $\oint_C f(z) \, dz$.

(a). $f(z) = \frac{z}{2z+3}$; C : the unit circle $|z| = 1$ $\oint_C f(z) \, dz = 0$

(b). $f(z) = \frac{z}{2z+3}$; C : the circle $|z| = 2$ $-\frac{3\pi}{2}i$

(c). $f(z) = \tan z$; C : the unit circle $|z| = 1$ $\oint_C f(z) \, dz = 0$

6. Given $\oint_C \frac{5}{z-1-i} \, dz$ and $C : x^4 + y^4 = 16$.

(a). Explain why $\oint_C \frac{5}{z-1-i} \, dz \neq 0$ There is a singularity at $z_0 = 1 + i$ which is interior to C

(b). Over which of the following contours will the integral give you the same answer.

- (i). $|z| = 2$ same (ii). $|z| = 1/2$ not same (iii). $|z - 1 - i| = 1$ same
 (iv). square with vertices at $z = 0, 2, 2 + 2i, 2i$ same

7. Use the Cauchy Integral formula, when appropriate, to evaluate

(a). $\oint_C \frac{z^2 - 3z + 4i}{z + 2i} dz$; $C : |z| = 3$ -8\pi(4 + i)

(b). $\oint_C \frac{z^2}{z^2 + 4} dz$; $C : (i) |z - i| = 2$ (ii) $|z + 2i| = 1$ (i) -2\pi (ii) 2\pi

(c). $\oint_C \frac{2z + 5}{z^2 - 2z} dz$; $C : (i) |z| = 1/2$ (ii) $|z + 1| = 2$ (iii) $|z - 3| = 2$ (iv) $|z + 2i| = 1$ (i)(ii) -5\pi i (iii) 18\pi i (iv) 0

(d). $\oint_C \frac{2z^3 + 3z}{(z + 2)^3} dz$; $C : |z - i| = 4$ -24\pi i

8. Let C_R denote the quarter circle in quadrant I of $|z| = R (R > \sqrt{5})$, taken in the counterclockwise direction.

(a). Find an upper bound on $\left| \int_{C_R} \frac{3z^2 - 2}{z^2 + 5z^4} dz \right|$.

(b). Find the limit of the upper bound found in part (a) as $R \rightarrow \infty$.

9. Determine whether the following series converge or diverge. If it converges, find the sum.

(a). $\sum_{n=0}^{\infty} \left(\frac{\pi i}{3} \right)^n$

(b). $\sum_{n=0}^{\infty} \frac{3^{n+1}}{(1 - 2i)^{n-2}}$

10. Determine for which values of z the following series converges and find (and simplify) the sum.

$$\sum_{n=0}^{\infty} \frac{(2z + 6i)^n}{3^n}$$

11. Rewrite the following function as a geometric series and state where it converges. $f(z) = \frac{3}{3z - 4}$

12. Derive the Taylor Series for $\sinh(z)$ by

(a). Using the definition of Taylor Series.

(b). Using the definition $\sinh z = \frac{e^z - e^{-z}}{2}$ and the known Maclaurin Series for e^z .

(c). Using the identity $\sinh z = -i \sin(iz)$ and the known Maclaurin Series for $\sin z$.

13. Use a known Maclaurin Series to find a series expansion for $f(z) = \frac{3z^2}{4z^4 + 5}$ and state the values of z where it converges. Is the resulting series a Maclaurin Series?