1. (Preliminary Work) DEF The Gamma Function is defined as $\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}\,dy$ for $t > 0$.

(a). Use integration by parts once to rewrite the Gamma Function. [Remember you are integrating with respect to y.]

$$u = y^{t-1} \qquad dv = e^{-y}\,dy$$

$$du = \underline{\qquad\qquad}\,dy \qquad v = \underline{\qquad\qquad}$$

$$\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}\,dy = \underline{\qquad\qquad}\Big|_0^\infty + \int_0^\infty (t-1)y^{t-2}e^{-y}\,dy$$

$$= \lim_{y\to\infty}\left(-y^{t-1}e^{-y}\right) + \underline{\qquad\qquad} + (t-1)\int_0^\infty \underline{\qquad\qquad}$$

The limit goes to ______ since

exponentials decay much faster than any power grows.

$$= 0 + 0 + (t-1)\int_0^\infty y^{t-2}e^{-y}\,dy$$

$$= (t-1)\underbrace{\int_0^\infty y^{t-2}e^{-y}\,dy}_{\Gamma(\qquad)}$$

$$= (t-1)\Gamma(t-1)$$

i.e. $\Gamma(t) = (t-1)\Gamma(t-1)$

(b). Use the formula from part (a) to find and simplify $\Gamma(5)$ (i.e. $t = 5$).

$\Gamma(5) = 4\Gamma(4) = 4(3\Gamma(3)) = 4\cdot 3(\underline{\qquad\qquad}) = 4\cdot 3\cdot 2(\underline{\qquad\qquad}) = 4\cdot 3\cdot 2\cdot 1\cdot\Gamma(1) = \underline{\qquad\qquad}\cdot\Gamma(1)$

(c). Use the result of part(b) to guess the formula for $t = n$ a positive integer: $\Gamma(n) = \underline{\qquad\qquad\qquad}$

(d). Evaluate the integral to determine $\Gamma(1)$.

$$\Gamma(1) = \int_0^\infty y^0 e^{-y}\,dy = \int_0^\infty e^{-y}\,dy =$$

(e). Combine the results of parts (c) and (d): $\boxed{\Gamma(n) = \qquad\qquad}$ We'll use this formula later.

Poisson Distribution with mean λ (i.e. λ occurrences per unit interval): $f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \ldots$

$\Longrightarrow$

Exponential Distribution with parameter $\theta = \dfrac{1}{\lambda}$ (i.e. mean time until the 1^{st} occurrence): $f(x) = \dfrac{1}{\theta} e^{-x/\theta}, \quad x \geq 0$

Suppose we now want to consider the random variable X that represents the waiting time until the α^{th} occurrence.

2. Derive the c.d.f for X.

$F(x) = P(X \leq x)$

$= 1 - P(X > x)$ i.e. The _________ occurrence happens after time ________ .

$= 1 - P($less than ________ occurrences in the interval ___________$)$

$= 1 - P(Y < \alpha)$ where Y is the number of occurrences in the interval $[0, x]$.

So Y is _____________ with parameter _________ .

$= 1 - [P(Y = 0) + P(Y = 1) + P(Y = 2) + \ldots + P(Y =$ ___________$)]$

$$= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}$$

i.e. $F(x) = 1 - \displaystyle\sum_{k=0}^{\alpha-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}$

3. Use the c.d.f $F(x) = 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}$ to derive the p.d.f for X. Recall $f(x) = F'(x)$

Then since $F(x) = 1 - e^{-\lambda x} - e^{-\lambda x} \sum_{k=1}^{\alpha-1} \frac{(\lambda x)^k}{k!}$ Separated out the ____________ term and factored out $e^{-\lambda x}$ from the sum since it does not depend on ________ .

$$f(x) = F'(x) = 0 - e^{-\lambda x} \cdot (\underline{\qquad\qquad}) - \left(\underline{\qquad\qquad} \sum_{k=1}^{\alpha-1} \frac{k(\lambda x)^{k-1} \cdot \lambda}{k!} + \sum_{k=1}^{\alpha-1} \frac{(\lambda x)^k}{k!} \cdot e^{-\lambda x} \cdot (-\lambda) \right) \quad \text{[Prod. Rule]}$$

$$= \lambda e^{-\lambda x} - \left(\lambda e^{-\lambda x} \sum_{k=1}^{\alpha-1} \frac{k(\lambda x)^{k-1}}{k!} - \lambda e^{-\lambda x} \sum_{k=1}^{\alpha-1} \frac{(\lambda x)^k}{k!} \right)$$

$$= \lambda e^{-\lambda x} - \lambda e^{-\lambda x} \sum_{k=1}^{\alpha-1} \left[\frac{k(\lambda x)^{k-1}}{k!} - \qquad\qquad \right] \quad \text{but } \frac{k}{k!} = \frac{k}{\underline{\qquad\qquad} \cdot k} = \frac{1}{\underline{\qquad\qquad}}$$

$$= \lambda e^{-\lambda x} - \lambda e^{-\lambda x} \sum_{k=1}^{\alpha-1} \left[\frac{(\lambda x)^{k-1}}{(k-1)!} - \frac{(\lambda x)^k}{k!} \right]$$

$$= \lambda e^{-\lambda x} - \lambda e^{-\lambda x} \left[\left(\frac{(\lambda x)^0}{0!} - \frac{(\lambda x)^1}{1!} \right) + \left(\qquad\qquad \right) + \left(\qquad\qquad \right) + \cdots \right.$$

$$\left. \cdots + \left(\frac{(\lambda x)^{\alpha-2}}{(\alpha-2)!} - \frac{(\lambda x)^{\alpha-1}}{(\alpha-1)!} \right) \right]$$

$$= \lambda e^{-\lambda x} - \lambda e^{-\lambda x} \left[1 - \qquad\qquad \right]$$

Cancel out terms above.

$$= \lambda e^{-\lambda x} - \underline{\qquad\qquad} + \lambda e^{-\lambda x} \frac{(\lambda x)^{\alpha-1}}{(\alpha-1)!}$$

$$= \underline{\qquad\qquad\qquad}$$

i.e. The p.d.f. of X is given by $f(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{\alpha-1}}{(\alpha-1)!}$.

4. Let $\lambda = \dfrac{1}{\theta}$ and also use the formula at the bottom of page 1 for the Gamma Function rewrite the p.d.f $f(x)$.

$$
\begin{aligned}
f(x) &= \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{(\alpha-1)!} \\
&= \frac{\frac{1}{\theta}e^{-x/\theta}\left(\frac{1}{\theta}x\right)^{\alpha-1}}{\qquad} \\
&= \frac{\frac{1}{\theta}e^{-x/\theta}\left(\qquad\right)^{\alpha-1} x^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{e^{-x/\theta}\left(\frac{1}{\theta}\right)^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)}
\end{aligned}
$$

But $\dfrac{1}{\theta}\cdot\left(\dfrac{1}{\theta}\right)^{\alpha-1} =$ __________

i.e.

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}}\, x^{\alpha-1}e^{-x/\theta}, \quad 0 \le x < \infty$$

Gamma Distribution with parameters α and $\theta = \dfrac{1}{\lambda}$

gives the waiting time until the α^{th} occurrence in a Poisson Process. Denoted $\Gamma(\alpha, \theta)$.

Note:

- Derived for α an integer, but also has applications for non-integer α.
- α is the shape parameter
- θ is the scale parameter

[Dr. Crawford will show typical sketches of $f(x)$.]

For the Gamma Distribution, You will derive these formulas in the homework

$\mu = \alpha\theta$ $\sigma^2 = \alpha\theta^2$

$$M(t) = \frac{1}{(1-\theta t)^{\alpha}}, \qquad t < \frac{1}{\theta}$$

Ex The number of times that a network computer fails has a Poisson distribution with a mean of once per month (30 days).

(a). Find the probability that it will be at least 3 months until the 2^{nd} failure. (Use months)

(b). What if you used days (30 days = 1 month)

(c). Find μ and σ^2

Homework: Section 3.2, p. 109: #7, 9, 10, 12, 13, 16, 17[Hint: Two-parter with binomial distribution], 19[Set up the integrals for E(Profit) and explain how you would solve it – but do not actually solve it.]