MTH 421 Exam 2 Key 

1. Both get through $\Rightarrow$ 2 successes in 4 tries } Negative Binomial $r = 2$

all busy 60% of time $\Rightarrow$ probability of getting through $p = 0.4$

$$P(X=4) = \binom{4-1}{2-1}(.4)^2(.6)^{4-2} = \binom{3}{1}(.4)^2(.6)^2 = 3(.4)^2(.6)^2$$
$$\approx \boxed{0.1728}$$

2. $Y \sim$ Poisson w/ $\lambda = 2$

Profit $X = 50 - 2Y - Y^2$

$$E(X) = E[50 - 2Y - Y^2] = E[50] - 2E[Y] - \underbrace{E[Y^2]}$$
$$= 50 - 2(2) - 6$$ See work below
$$= \boxed{40}$$

Note: For Poisson $\mu = E[Y] = \lambda = 2$

$$\sigma^2 = E[Y^2] - (E[Y])^2 = \lambda = 2$$
$$\Rightarrow E[Y^2] - (2)^2 = 2$$
$$E[Y^2] = 6$$

3. $f(y) = \begin{cases} cy^2 + y, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

(a) $\int_0^1 cy^2 + y\,dy = \frac{c}{3}y^3 + \frac{1}{2}y^2 \Big|_0^1 = \frac{c}{3} + \frac{1}{2} - (0)$

$$= \underbrace{\frac{c}{3} + \frac{1}{2} = 1}_{\frac{c}{3} = \frac{1}{2}} \Rightarrow \boxed{C = \frac{3}{2}}$$

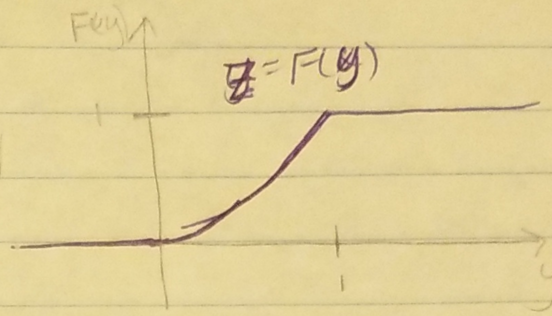
(b) $F(y) = \underbrace{\int_0^y f(t)\,dt}_{\text{for } 0 \le y \le 1} = \int_0^y \frac{3}{2}t^2 + t\,dt = \frac{1}{2}t^3 + \frac{1}{2}t^2 \Big|_0^y = \frac{1}{2}y^3 + \frac{1}{2}y^2$

For $y > 1$: $F(y) = \int_0^y f(t)\,dt = \underbrace{\int_0^1 f(t)\,dt}_{1} + \int_1^y f(t)\,dt$

$$= 1 + \int_1^y \underbrace{0}_{0}\,dt$$
$$= 1$$

3(b) Cont.

$$F(y)=\begin{cases}0, & y<0\\ \frac{1}{2}y^3+\frac{1}{2}y^2, & 0\le y\le 1\\ 1, & y>1\end{cases}$$




(c) 15 min = $\frac{1}{4}$ hour

30 min = $\frac{1}{2}$ hour

$$P(Y>\tfrac{1}{2} \mid Y>\tfrac{1}{4}) = \frac{P(Y>\frac{1}{2} \text{ and } Y>\frac{1}{4})}{P(Y>\frac{1}{4})} = \frac{P(Y>\frac{1}{2})}{P(Y>\frac{1}{4})}$$

$$= \frac{1-P(Y\le\frac{1}{2})}{1-P(Y\le\frac{1}{4})} = \frac{1-F(\frac{1}{2})}{1-F(\frac{1}{4})} = \frac{1-0.1875}{1-0.0390625}$$

$$= \boxed{0.8455}$$

4. $N(500, 25^2)$ $\mu = 500$ $\sigma = 25 \Rightarrow \sigma^2 = 25^2$

(a) $P(X>550) = \text{normalcdf}(550, 9999, 500, 25)$

$$\approx \boxed{0.02275}$$

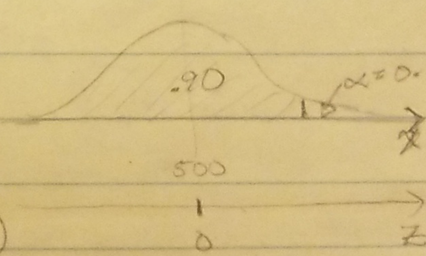
OR $P(X>550) = P\left(\frac{X-500}{25} > \frac{550-500}{25}\right) = P(Z>2)$

$$= 1-\Phi(2) = 1-\underbrace{0.9772}_{\text{Table Va}} \approx \boxed{.0228}$$

(b) Find $x$, s.t. $P(X>x) = 0.1$

$1-P(X\le x) = 0.1$

$P(X\le x) = 0.9$




$x = \text{InvNorm}(0.9, 500, 25)$

$\approx \boxed{\$532.04}$

OR

$\Phi(x) = 0.9$

$z = \underbrace{1.282}_{\text{Bottom of Table Va}}$

$1.282 = \frac{x-500}{25}$

$\Rightarrow x = 500 + 1.282(25)$

$= \boxed{\$532.05}$

5. $Y \sim e^X$ where $X \sim N(10, 1)$

(a) $F(y) = P(Y \le y) = P(e^X \le y) = P(\ln e^X \le \ln y) = P(X \le \ln y)$

$$\Rightarrow P(X \le \ln y) = P\left(\frac{X-10}{1} \le \frac{\ln y - 10}{1}\right)$$

$$= P(Z \le \ln y - 10)$$

$$F(y) = \boxed{\Phi(\ln y - 10)}$$ where $\Phi(z)$ is the cdf for $N(0,1)$

(b) $P(10{,}000 < Y < 20{,}000) = F(20{,}000) - F(10{,}000)$

$$= \Phi(\ln 20000 - 10) - \Phi(\ln 10{,}000 - 10)$$

$$\approx \Phi(-.0965) - \Phi(-.7897)$$

$$\approx 0.46156 - 0.21486$$ ← using calculator normalcdf

$$\approx \boxed{0.2467}$$

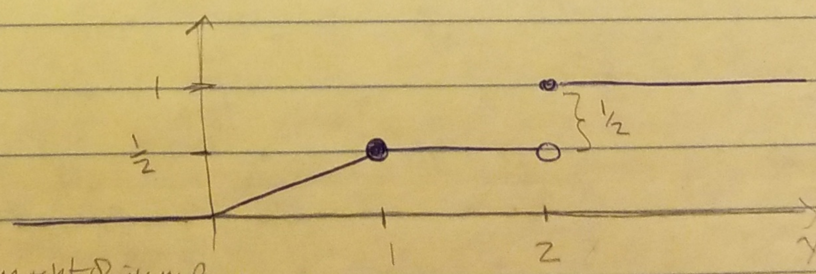
OR $\Phi(-.0965) - \Phi(-.7897) \approx \Phi(-.10) - \Phi(-.79)$

$$= 1 - \Phi(.10) - (1 - \Phi(.79))$$ Table Va

$$= \Phi(.79) - \Phi(.10)$$

$$\approx 0.7852 - 0.5398 \approx \boxed{0.2454}$$

6. Typo → $F(x) = \begin{cases} 0, & x < 0 \\ x/2, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$




a) $P(X = 2) = \boxed{\frac{1}{2}}$ height of jump

b) $f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & 0 \le x < 1 \\ 0 & 1 \le x < 2 \\ 1/2 & X = 2 \\ 0, & x > 2 \end{cases}$

$E[X] = \int_0^1 x \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}$ ← Mixed

$$= \frac{1}{4}x^2 \Big|_0^1 + 1$$

$$= \frac{1}{4} - 0 + 1 = \boxed{\frac{5}{4}}$$

7. $f(x,y) = \frac{xy^2}{22}$ $(1,1), (1,2), (1,3), (2,2)$

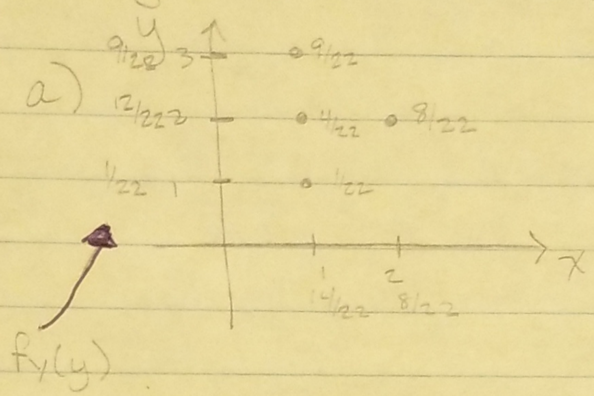
$f(1,1) = \frac{1}{22}$

$f(1,2) = \frac{1\cdot 4}{22} = \frac{4}{22} = \frac{2}{11}$

$f(1,3) = \frac{1\cdot 9}{22} = \frac{9}{22}$

$f(2,2) = \frac{2\cdot 4}{22} = \frac{8}{22} = \frac{4}{11}$

a)




(b) $\boxed{\text{No}}$, the space is not rectangular

OR Observe

$f(1,1) = \frac{1}{22}$ $f(x,y) \neq f_X(x) f_Y(y)$

but $f_X(1) f_Y(1) = \frac{14}{22} \cdot \frac{1}{22} = \frac{14}{22}$

(c) $P(X=Y) = P(X=1, Y=1) + P(X=2, Y=2)$

$= f(1,1) + f(2,2)$

$= \frac{1}{22} + \frac{8}{22} = \boxed{\frac{9}{22}}$

(d) $\mathrm{Cov}(X,Y) = E[XY] - \mu_X \mu_Y$

$E[XY] = \sum\sum xy f(x,y) = 1\cdot 1 f(1,1) + 1\cdot 2 f(1,2) + 1\cdot 3 f(1,3) + 2\cdot 2 f(2,2)$

$= 1\left(\frac{1}{22}\right) + 2\cdot\frac{4}{22} + 3\cdot\frac{9}{22} + 4\cdot\frac{8}{22} = \frac{1+8+27+32}{22} = \frac{68}{22} = \frac{34}{11}$

$\mu_X = \sum_x \sum_y x \cdot f(x,y) = 1\cdot f(1,1) + 1\cdot f(1,2) + 1\cdot f(1,3) + 2\cdot f(2,2)$

$= 1\cdot\frac{1}{22} + 1\cdot\frac{4}{22} + 1\cdot\frac{9}{22} + 2\cdot\frac{8}{22} = \frac{1+4+9+16}{22} = \frac{30}{22} = \frac{15}{11}$

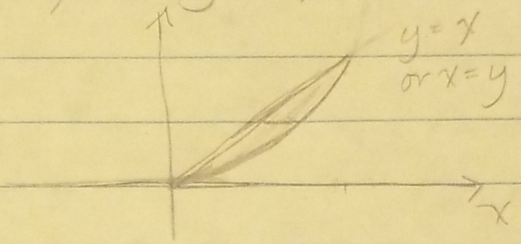
$\mu_Y = \sum\sum y f(x,y) = 1\cdot f(1,1) + 2\cdot f(1,2) + 3\cdot f(1,3) + 2\cdot f(2,2)$

$= 1\cdot\frac{1}{22} + 2\cdot\frac{4}{22} + 3\cdot\frac{9}{22} + 2\cdot\frac{8}{22} = \frac{11+8+27+16}{22} = \frac{52}{22}$

$= \frac{26}{11}$

$\mathrm{Cov}(X,Y) = \frac{34}{11} - \frac{15}{11}\cdot\frac{26}{11} = \frac{34\cdot 11 - 15\cdot 26}{11^2} = \boxed{\frac{-16}{121} \approx -0.1322}$

8. $f(x,y) = cxy \qquad 0 \le x \le 1,\ x^2 \le y \le x \qquad y = x^2 \text{ or } x = \sqrt{y}$




$1 = \int_0^1 \int_{x^2}^{x} cxy \, dy \, dx$

Inner Integral:

$\int_{x^2}^{x} \frac{1}{2} c \, xy^2 \Big|_{x^2}^{x}$

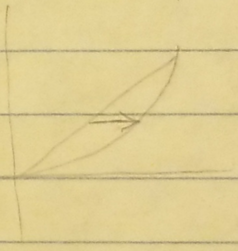
$= \frac{1}{2} cx \left[ x^2 - (x^2)^2 \right] = \frac{1}{2} c (x^3 - x^5)$

$\Rightarrow \int_0^1 \frac{1}{2} c (x^3 - x^5) \, dx = \frac{1}{2} c \left( \frac{1}{4} x^4 - \frac{1}{6} x^6 \right) \Big|_0^1$

$= \frac{1}{2} c \left( \frac{1}{4} - \frac{1}{6} \right) - (0)$

$= \frac{1}{2} c \left( \frac{3-2}{12} \right) = \frac{1}{2} c \left( \frac{1}{12} \right) = \frac{c}{24}$

$\Rightarrow \frac{c}{24} = 1 \Rightarrow \boxed{c = 24}$



(b) $\mu_Y = \int_0^1 \int_{x^2}^{x} y \cdot f(x,y) \, dy \, dx$

$= \int_0^1 \int_{x^2}^{x} 24xy^2 \, dy \, dx$

$= \int_0^1 8xy^3 \Big|_{x^2}^{x} dx$

$= \int_0^1 8x(x^3 - (x^2)^3) \, dx$

$= \int_0^1 8x^4 - 8x^7 \, dx$

$= \frac{8}{5} x^5 - x^8 \Big|_0^1$

$= \frac{8}{5} - 1 - (0)$

$\boxed{\frac{3}{5}}$

OR $\int_0^1 \int_y^{\sqrt{y}} y \cdot 24xy \, dx \, dy$

$= \int_0^1 \int_y^{\sqrt{y}} 24xy^2 \, dx \, dy$

$= \int_0^1 12x^2y^2 \Big|_y^{\sqrt{y}} dy$

$= \int_0^1 12y^2 \left[ (\sqrt{y})^2 - y^2 \right] dy$

$= \int_0^1 12y^2 (y - y^2) \, dy$

$= \int_0^1 12y^3 - 12y^4 \, dy$

$= 3y^4 - \frac{12}{5} y^5 \Big|_0^1$

$= 3 - \frac{12}{5} - (0)$

$= \boxed{\frac{3}{5}}$ ✓