1. THEOREM If f is differentiable at a point a, then f is continuous at a.

PROOF:

Let f be differentiable at a. [Show that f is continuous at a]

Then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is a finite number. [Show $\lim_{x \to a} f(x) =$ __________]

["Trick"] Write $f(x) = (x - a) \cdot \frac{f(x) - f(a)}{x - a} + f(a)$ ⟵ Convince yourself this is true.

Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[(x - a) \cdot \frac{f(x) - f(a)}{x - a} + f(a) \right]$$

$= \lim_{x \to a} (x - a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a}$ __________ by limit laws since they are all finite.

$= 0 \cdot$ __________ $+ f(a)$ by evaluating the limits (using limit laws and the definition of derivative).

$=$ __________

i.e. ______________ $= f(a)$

Therefore, by definition, $f(x)$ is ________________ at a. ■

2. The converse is not true: If $f(x)$ is continuous at a, then it may or may not be differentiable at a.

(a). Sketch a picture of $f(x) = |x|$ and note that it is continuous at $x = 0$.

However, $f(x) = |x|$ is not differentiable at $x = 0$. In terms of the picture and tangent lines, explain in your own words why it can't be differentiable there. [Note:"Because it has a sharp corner" is not a sufficient explanation. Basically you need to explain why corners are not differentiable.]

(b). Use the limit definition of derivative to show that $f(x) = |x|$ is not differentiable at $x = 0$. [Hint: Look at the one sided limits].