<u>DEF</u> (Limit of a function) Let $S \subset \mathbb{R}$ and let $a \in \mathbb{R}$ or symbol $+\infty$ or $-\infty$. Also let $L \in \mathbb{R}$ or symbol $+\infty$ or $-\infty$. Then for a function defined on S,

$$\lim_{x \to a^S} f(x) = L$$

means that for each sequence (x_n) in S with $\lim x_n = a$, we have $\lim_{n \to \infty} f(x_n) = L$.

We say, "the limit as x approaches a along S is L."

Notation: Let J be an open interval containing a. Then $S = J \backslash \{a\} = \{x \in J \mid x \neq a\}$

For example, $J = (c, b)$ where $c < a < b$ Then if $S = J \backslash \{a\} = (c, a) \cup (a, b)$

<u>DEF</u> Standard Limit Definitions:

- <u>Two-sided limit</u>: $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some set $S = J \backslash \{a\}$ where J is an open interval containing a.

  We say, "the limit of $f(x)$ as x approaches a is L."

- <u>Right-hand limit</u>: $\lim_{x \to a^+} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some set $S = (a, b)$

- <u>Left-hand limit</u>: $\lim_{x \to a^-} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some set $S = (c, a)$

- <u>Limit at infinity</u>: $\lim_{x \to \infty} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some set $S = (c, \infty)$

  or $\lim_{x \to -\infty} f(x) = L$ if $\lim_{x \to a^S} f(x) = L$ for some set $S = (-\infty, b)$

Ex Given that $f(x) = \begin{cases} x - 1, & x \geq 2 \\ x + 3, & x < 2 \end{cases}$, prove $\lim_{x \to 2^-} f(x) = 5$.

Ex Prove $\lim_{x \to \infty} \dfrac{1}{x^2 + 1} = 0$.

Note: f need not be defined at a. Even if it is $f(a)$ need not equal $\lim_{x\to a} f(x)$. [sketch]

THEOREM : If $f(x) = g(x)$ for all $x \in S = J \backslash \{a\}$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

PROOF: Obvious from definitions.

THEOREM : $\lim_{x\to a} f(x) = f(a)$ iff f is continuous at a. [See book for proof.]

Note: This theorem justifies using direct substitution to evaluate limits.

Ex: $\lim_{x\to 3} x^2$

Ex: $\lim_{x\to 3} \frac{9 - x^2}{x - 3}$

LIMIT LAWS for ***finite*** limits

Let f_1 and f_2 be functions with the following limits

$$\lim_{x\to a} f_1(x) = L_1 \quad \text{and} \quad \lim_{x\to a} f_2(x) = L_2 \qquad \text{where } L_1, L_2 \in \mathbb{R} \text{ (i.e. finite)},$$

then

1. $\lim_{x\to a^S} [f_1(x) \pm f_2(x)] = L_1 \pm L_2$

2. $\lim_{x\to a^S} [f_1(x) \cdot f_2(x)] = L_1 \cdot L_2$

3. $\lim_{x\to a^S} \left[\frac{f_1(x)}{f_2(x)}\right] = \frac{L_1}{L_2}$ provided $L_2 \neq 0$ and $f_2(x) \neq 0$ for $x \in S$.

PROOF: Follows easily from Limit Laws for sequences in Section 9

For Composite Functions:

4. Let f be a function with $\lim_{x\to a^S} f(x) = L \in \mathbb{R}$ (finite). Let g be a function defined on $\{f(x) \mid x \in S\} \cup L$.
If g is continuous at L, then $\lim_{x\to a^S} (g \circ f)(x) = \lim_{x\to a^S} g(f(x)) =$ ___$g(L)$___

[See book for proof.]

Ex Evaluate $\lim_{x\to 4} e^{x^2}$

THEOREM ($\epsilon - \delta$ property for limits of functions)

Let f be defined on $S \subseteq \mathbb{R}$ and let x_n be in S with $\underline{\lim x_n = a}$ $\in \mathbb{R}$, and let $L \in \mathbb{R}$. Then $\lim_{x \to a} f(x) = L$ iff for each $\epsilon > 0, \exists$ $\underline{\delta > 0}$, such that for $x \in S$ and $|x - a| < \delta$, we have $\underline{|f(x) - L| < \epsilon}$.

Note: Similar property for one-sided limits and limits at infinity – see book

THEOREM Let f be a function defined on $S = J \backslash \{a\}$ for some open interval J containing a. Then $\lim_{x \to a} f(x)$ exists and equals L iff $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

PROOF: Uses the $\epsilon - \delta$ properties

Ex: Heaviside function $H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$

(a). Sketch $H(x)$ and determine limits as $x \to 0^-, 0^+, 0$.

(b). Prove the assertions in part (a) (using the sequence definition(s)).

Homework: Section 20: #1, 2, 5^*, 6^*, 9, 10^* [11-14 Use Limit Laws/Theorems]
*Use sequence definition(s).