Completeness Axiom: Every nonempty subset S of $\mathbb{R}$ that is bounded above has a real supremum.

i.e ______________________________ .

Complete the proof below for the Corollary to the Completeness Axiom

Corollary: Every nonempty subset S of $\mathbb{R}$ that is bounded below has a real infimum.

PROOF:

Let S be a nonempty subset of $\mathbb{R}$ and ____________________ . [Show $\inf S$ exists.]

Define $T = \{t = -s | s \in S\}$.

Since S is bounded below, $\exists m \in \mathbb{R}$, such that ____________ $\forall s \in S$.

Then $-s \leq -m \; \forall s \in S$.

Thus ____________ $\forall t \in T$.

In other words, T is bounded above by __________ .

Now by the Completeness Axiom, _________________ .

[To complete the proof we will show that $-\sup T = \inf S$.
In order to show this, we must verify the definition of infimum by showing

(i): $-\sup T$ is a _________________ of S and *(ii):* $-\sup T$ is $\geq$ all other lower bounds of S.]

By definition of supremum,

______________ $\forall t \in T$

$\sup T \geq -s \; \forall s \in S$

_____________ $\leq$ _______ $\forall s \in S$

Thus, $-\sup T$ is ___________________ for S. [Thus, we have shown *(i)*]

Let c be a lower bound for S. [Show that _________________ .]

Then ___________ $\forall s \in S$.

Then $-s \leq -c \; \forall s \in S$, which implies that

$t \leq -c \; \forall t \in T$,

which implies that $-c$ is ____________________ for T.

Thus, $\sup T \leq -c$ by ______________________________ .

Therefore, _________________ . In other words, $-\sup T$ is the greatest lower bound. [i.e. We have shown *(ii)*.]

Thus, by definition of infimum, ____________________ .

Therefore, $\inf S \in \mathbb{R}$ exists. ■

Complete the proof below for the Archimedean Property. [Note: This property holds for both $\mathbb{R}$ and $\mathbb{Q}$.]

Archimedean Property: If $a > 0$ and $b > 0$, then $\exists n \in \mathbb{N}$ such that $na > b$

PROOF:

Let $a > 0, b > 0$ and BWOC suppose ____________________ . [Hint: Don't forget to negate "there exists"]

Let $S = \{na \mid n \in \mathbb{N}\}$.

Then ________ is an upper bound for S,

and by the Completeness Axiom, __________________ .

Since $a > 0$, then $-a < 0$.

Then adding $\sup S$ to both sides $\Rightarrow$ ________________ $< \sup S$.

Thus $\sup S - a$ is not an upper bound for S because ____ .

Hence, there must be an ***element of*** S that is ____________________________ .

That is, $\exists ma \in S$ such that $ma > \sup S - a$.

Then $ma + a$______________.

$\Rightarrow (m+1)a > \sup S$. (*)

But $(m+1) \in \mathbb{N}$,

so $(m+1)a \in$ ________. $\rightarrow\!\leftarrow$

This is a contradiction, since an element of S cannot be ________________________ [see (*)] or $\sup S$ would fail to be an upper bound.

Therefore, $\exists n \in \mathbb{N}$, s.t. $na > b$. ■

1. Prove the following 2 corollaries to the Archimedean Property.
[Hint: Corollaries are direct results of a"main" theorem... so use the main theorem. Don't redo the proof of the main theorem.]

(a). If $a > 0$, then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < a$.

(b). If $b > 0$, then $\exists n \in \mathbb{N}$ such that $b < n$.

2. Section 4, p. 26: #1-4(a, e, i , m, q, u, w), 7(b), 8, 10, [11, 14 if finished Denseness of $\mathbb{Q}$]

3. Prove the following:
Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. If $x < \sup(A)$, then $\exists a \in A$ such that $x < a < \sup(A)$.