Def A **field** is a set S that satisfies the following field properties:

Field Properties

Let S be a set such that 0 and 1 are in S and the operations of addition $(+)$ and multiplication $(\cdot)$ are defined on S. Then for $a, b, c \in S$ consider the following properties:

Closure	A0.	$a + b \in S$	M0.	$ab \in S$
Associative	A1.	$a + (b + c) = (a + b) + c$	M1.	$a(bc) = (ab)c$
Commutative	A2.	$a + b = b + a$	M2.	$ab = ba$
Identity	A3.	$a + 0 = a$	M3.	$a \cdot 1 = a$
Inverse	A4.	$\forall a, \exists -a \in S$ s.t. $a + (-a) = 0$	M4.	$\forall a \neq 0, \exists a^{-1} \in S$ s.t. $a \cdot a^{-1} = 1$

Distributive DL. $a(b + c) = ab + ac$

Determine which of the following sets are fields. If it is not a field, indicate which properties fail to hold.

- Natural Numbers $\mathbb{N}$ $0 \notin \mathbb{N} \Rightarrow$ A3; $-a \notin \mathbb{N} \Rightarrow$ A4; $a^{-1} \notin \mathbb{N} \Rightarrow$ M4
- Integers $\mathbb{Z}$ $a^{-1} \notin \mathbb{Z} \Rightarrow$ M4
- Rational Numbers $\mathbb{Q}$ Field
- Real Numbers $\mathbb{R}$ Field
  [Even though we haven't formally defined $\mathbb{R}$, answer based on what you know about $\mathbb{R}$.]

Def If a field (as defined above) also satisfies the following properties, it is called an **ordered field**.

O1. $\forall a, b$ either $a \leq b$ or $b \leq a$

O2. If $a \leq b$ and $b \leq a$, then $a = b$

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$ (Transitive)

O4. If $a \leq b$, then $a + c \leq b + c$

O5. If $a \leq b$ and $0 \leq c$ (nonnegative), then $ac \leq bc$

Are the fields of Rational Numbers $\mathbb{Q}$ and Real Numbers $\mathbb{R}$ also ordered fields? Yes

So far

Theorem: Consequences of Field Properties Let a, b, c be elements of a field F.

(i). If $a + c = b + c$, then $a = b$.

(ii). $a \cdot 0 = 0, \ \forall a$.

(iii). $(-a)(b) = -(ab) = -ab, \ \forall a, b$.

(iv). $(-a)(-b) = ab, \ \forall a, b$.

(v). If $ac = bc$ and $c \neq 0$, then $a = b$.

(vi). If $ab = 0$, then $a = 0$ or $b = 0$.

Theorem: Consequences of Ordered Field Properties Let a, b, c be elements of an ordered field F.

(i). If $a \leq b$, then $-b \leq -a$.

(ii). If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

(iii). If $0 \leq a$ and $0 \leq b$, then $0 \leq ab$.

(iv). $0 \leq a^2, \ \forall a$.

(v). $0 < 1$

(vi). If $0 < a$, then $0 < a^{-1}$.

(vii). If $0 < a < b$, then $0 < b^{-1} < a^{-1}$

<u>Ex</u>: Prove part (i) of the Consequences of Field Properties using only the Field Properties.

<u>Ex</u>: Prove part (i) of the Consequences of Field Properties using only the Field Properties and Ordered Field Properties.

Fill in the blanks in the following proof for part *(ii)* of the Consequences of Field Properties.

1. Let F be a field and let $a, b, c \in F$. Prove: $a \cdot 0 = 0$ for all a.

PROOF

Let $a \in F$. [Show $\underline{\quad a \cdot 0 = 0 \quad}$]

Consider

$$
\begin{aligned}
0 &= 0 \\
0 + 0 &= 0 && \text{by } \underline{\quad \text{A3} \quad} \\
a \cdot (0 + 0) &= \underline{\quad a \cdot 0 \quad} && \text{by multiplying both sides by } a \\
a \cdot 0 + \underline{\quad a \cdot 0 \quad} &= a \cdot 0 && \text{by } \underline{\quad \text{DL} \quad} \\
a \cdot 0 + a \cdot 0 &= a \cdot 0 + 0 && \text{by } \underline{\quad \text{A3} \quad}
\end{aligned}
$$

Therefore, $\underline{\quad a \cdot 0 = 0 \quad}$ ■ (by part $\underline{\quad \textit{(i)} \text{ of CoFP} \quad}$: $a + c = b + c \Rightarrow \underline{\quad a = b \quad}$).

[Note: You will prove the remainder (parts *iii-vi*) of the Consequences of Field Properties as homework (see prob. #3). However, in order to complete #2 in class, assume that you have proven them. You can also use part *(i)* of Consequences of *Ordered* Field Properties since we already proved it in class.

Fill in the blanks for the following proof for part *(ii)* of Consequences of *Ordered* Field Properties.

2. Let F be an ordered field and let $a, b, c \in F$. Prove: If $a \leq b$ and $c \leq 0$, then $bc \leq ac$.

PROOF

Let $a \leq b$ and $\underline{\quad c \leq 0 \quad}$. [Show that $bc \leq ac$.]

$\Rightarrow 0 \leq -c$ by $\underline{\quad \text{CoOFP part } \textit{(i)} \quad}$

Now we have $a \leq b$ and $0 \leq -c$, so $\underline{\quad a(-c) \quad} \leq \underline{\quad b(-c) \quad}$ by O5

$\Rightarrow -ac \leq -bc$ by CoFP part $\underline{\quad \textit{(iii)} \quad}$ and M2

Therefore, $\underline{\quad bc \quad} \leq ac$ ■ by $\underline{\quad \text{CoOFP part } \textit{(i)} \quad}$.

3. Homework: Prove the remaining parts of the theorem Consequences of Field Properties.
Remember, you can only use previously proven parts.

4. Homework: Prove the remaining parts of the theorem Consequences of *Ordered* Field Properties.
Remember, you can only use previously proven parts.

[Note: If you eventually use the book to see how they prove some of them, rewrite the proof in your own style and justify the steps with the appropriate axioms, properties, previous theorems, etc.]