Name: Key **Quiz 1**

Math 381 Advanced Calculus – Crawford **20 September 2019**

Books, notes (in any form), and calculators are not allowed. You may use a sheet of Field Properties and their Consequences. *Show all your work.* Good Luck!

1. (6 pts) Using only the Field Properties and Consequences of Field Properties, prove the following. [Clearly justify each step by indicating which properties you use.]

Let $a \in F$, where F is a field. If $a \neq 0$, then

(a). $a^{-1} \neq 0$ [redacted]

Since $a \neq 0$, then $a^{-1} \in F$ exists s.t. $\overbrace{a \cdot a^{-1} = 1}^{(*)}$ by M4.

BWOC, suppose $a^{-1} = 0$.

Then $a \cdot a^{-1} = a \cdot 0 = 0$ by CoFP(ii) $\rightarrow$ ✱ (since $a a^{-1} = 1$)

$\therefore a^{-1} \neq 0$.

(b). $(a^{-1})^{-1} = a$.

Since $a^{-1} \neq 0$ by part (a)

then $(a^{-1})^{-1} \in F$ exists s.t. $(a^{-1})(a^{-1})^{-1} = 1$

But $a \cdot a^{-1} = 1$ from $(*)$.

Equating these two expressions and using the Commutative law,

$a \cdot a^{-1} = (a^{-1})^{-1} \cdot a^{-1}$. Then since $a^{-1} \neq 0$, $a = (a^{-1})^{-1}$ by CoFP(v).

2. (4 pts) Use induction to prove the following:

If $0 < x < y$, then $x^n < y^n$ for all $n \in \mathbb{N}$.

Let $0 < x < y$

Basis $n = 1$: $x < y$ true by the given hypotheses.

Induction: Assume true for $n = k$, ie $x^k < y^k$ [Show true for $n = k+1$, ie $x^{k+1} < y^{k+1}$]

$$\begin{aligned} x^{k+1} &= x^k \cdot x \\ &< y^k \cdot y \quad \text{since } x<y \text{ (given) and } x^k < y^k \text{ by the Induction Assumption} \\ &= y^{k+1} \end{aligned}$$

ie $x^{k+1} < y^{k+1}$

Since it is true for $n = k+1$, then by Mathematical Induction $x^n < y^n$ for all $n \in \mathbb{N}$.

3. (4 pts) For the following subsets of $\mathbb{R}$, state the minimum, maximum, infimum, and supremum, if they exist. If it does not exist, clearly state this.

Note: $\frac{n}{n+1} \to 1$

(a). $S = \left\{ \frac{n}{n+1} \,\middle|\, n \in \mathbb{N} \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\}$

max S = DNE
sup $S = 1$
min $S = \frac{1}{2}$
inf $S = \frac{1}{2}$

(b). $T = \{ r \in \mathbb{Q} \mid r^2 < 5 \}$ $\Rightarrow -\sqrt{5} < r < \sqrt{5}$

max T DNE
sup $T = \sqrt{5}$
min T DNE
inf $T = -\sqrt{5}$

4. (6 pts) Determine whether the following statements are TRUE or FALSE.
If it is FALSE, give a counterexample. If it is TRUE, no additional work needed.

T F $\forall a, b \in \mathbb{R}$, if $a < b$, then $|a| < |b|$.

Let $a = -4$, $b = 2$
Then $-4 < 2$, but $|-4| \not< |2|$
$4 \not< 2$

T F $\forall a, b \in \mathbb{R}$, $|a - b| \leq |a| + |b|$.

$|a-b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$

T F Let S be a nonempty bounded subset of $\mathbb{R}$. If $\sup S \in S$, then $\sup S = \max S$.