Name: ____________________________________ **Exam 2**

Math 381 Advanced Calculus – Crawford **15 November 2019**

Books and notes (in any form) are not allowed. There are 104 points on the exam, plus another 4 possible from the proof option. Please answer page 2 on this sheet. But do all of your other work including answers on separate paper and attach this sheet as a cover sheet. *Show all your work.* Good Luck!

1. (13 pts) Use an M-proof to show that $\lim \dfrac{4n^2-3}{n+3} = +\infty$.

2. (13 pts) Use an ϵ-proof to show that $s_n = 1 - \dfrac{2}{\sqrt{n}}$ is a Cauchy sequence.

3. (12 pts) Given $s_n = \left[\cos\left(\dfrac{n\pi}{3}\right)\right]^n$,

(a). Determine $\liminf s_n$ and $\limsup s_n$.

(b). Find the set of subsequential limits.

4. (16 pts) Determine whether the following series converge or diverge. Show all your work and clearly state any tests that you use. If the series is a converging geometric series, find the sum.

(a). $\displaystyle\sum_{n=1}^{\infty} \frac{ne^n}{(2n)!}$

(b). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

5. (26 pts) Prove ***two*** of the following.

BONUS: You may do (or attempt) all three options and each will be graded out of 12 points. Whichever two you score the highest on will be your base grade. Any points from the other problem will be cut in third and added to your base grade. (e.g. If you get 10 points, 5 points, and 3 points, then your score will be $10+5+\frac{3}{3}=16$)

(a). NOT NEW: Let s_n be a sequence in $\mathbb{R}$. If $\lim s_n = s \in \mathbb{R}$, then

(i) Prove that $\limsup s_n \leq s$ (*). [Hint: You may find the following lemma helpful: If $a \leq b+\epsilon$ for all $\epsilon > 0$, then $a \leq b$.]

(ii) A similar proof can show that $s \leq \liminf s_n$ (**). [Do not prove (**)].

Use (*) and (**) to complete the proof to show that $\liminf s_n = \limsup s_n = s$.

(b). NEW: Let (x_n) be a sequence of real numbers and let $a_n = x_n - x_{n+1}$ for all $n \in \mathbb{N}$.

(i) Prove that if the *sequence* (x_n) converges, then the *series* $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(x_n - x_{n+1})$ converges.

(ii) If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, what is the sum?

(c). NEW:

(i) Prove part(a) of The Limit Comparison Test Extension found on p. 2. [Clearly state what you are proving.]

(ii) Given an example of a series involving $\ln n$ that illustrates this extension.

6. (12 pts) The Limit Comparison Test can be extended to include a limit of 0 or a limit of ∞.

<u>The Limit Comparison Test Extension</u>
Suppose $\sum a_n$ and $\sum b_n$ are both *series* with *positive* terms.

Briefly explain why you need only consider the case where $\lim a_n = 0$ and $\lim b_n = 0$.

Fill in the blanks with "converges" or "diverges" for the extension below.

(a). If $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ and $\sum a_n$ __________________ , then $\sum b_n$ __________________ .

[***Briefly*** explain your answer.]

(b). If $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$ and $\sum a_n$ __________________ , then $\sum b_n$ __________________ .

[***Briefly*** explain your answer.]

7. (12 pts) Determine if the following statements are True or False. If the statement is false, give a counterexample or clearly explain why it is not possible.

In this section $s_k = \displaystyle\sum_{n=1}^{k} a_n$ is the k^{th} partial sum of the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$

T F If $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

T F If $\displaystyle\lim_{k\to\infty} s_k$ DNE, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

T F If (s_n) is unbounded above, then $\liminf s_n = \limsup s_n = +\infty$.

T F All convergent sequences are bounded.