MTH 381 – Exam 2 Key p1/

1. Show $\lim \frac{4n^2-3}{n+3} = +\infty$

Proof Let $M>0$ and choose $\boxed{N=4M}$.

Then, for $n>N$: $\frac{4n^2-3}{n+3} > \frac{n}{4}$ by sidework

$> \frac{N}{4}$ since $n>N$

$= \frac{4M}{4} = M$

ie $\frac{4n^2-3}{n+3} > M$ for all $n>N$.

$\therefore \lim \frac{4n^2-3}{n+3} = +\infty$ ∎

Sidework

[Want $\frac{4n^2-3}{n+3} > M$]

Numerator: $4n^2-3 \ge n^2 \ \forall n \ge 1$

Denom: $n+3 < n+3n = 4n$

$\Rightarrow \frac{1}{n+3} > 4n$

$\Rightarrow \frac{4n^2-3}{n+3} > \frac{n^2}{4n} = \frac{n}{4} \overset{set}{>} M$

$\Rightarrow n > 4M$

Choose $N = 4M$

2. Show that $s_n = 1 - \frac{2}{\sqrt{n}}$ is Cauchy

Proof Let $\varepsilon>0$ and choose $\boxed{N = \frac{4}{\varepsilon^2}}$

Then for $n, m > N$ we have

$|s_n - s_m| = |1-\frac{2}{\sqrt{n}} - (1-\frac{2}{\sqrt{m}})|$

$< \frac{2}{\sqrt{m}}$ by the sidework & $n>m$

$< \frac{2}{\sqrt{N}}$ since $m>N$

$= \frac{2}{\sqrt{4/\varepsilon^2}} = \frac{2}{2/\varepsilon} = \varepsilon$

ie $|s_n-s_m|<\varepsilon$ whenever $n>m\ge N$

$\therefore s_n$ is Cauchy. ∎

Sidework

[Want $|s_n-s_m|<\varepsilon$]

$|s_n - s_m| = |1-\frac{2}{\sqrt{n}}-(1-\frac{2}{\sqrt{m}})|$

$= \left|\frac{2}{\sqrt{m}} - \frac{2}{\sqrt{n}}\right|$

$= \left|\frac{2\sqrt{n} - 2\sqrt{m}}{\sqrt{m}\sqrt{n}}\right| = 2\left|\frac{\sqrt{n}-\sqrt{m}}{\sqrt{m}\sqrt{n}}\right|$ (*)

WLOG, let $n>m \Rightarrow \sqrt{n}>\sqrt{m}$

so (*) becomes $\frac{2(\sqrt{n}-\sqrt{m})}{\sqrt{m}\sqrt{n}}$

$< \frac{2\sqrt{n}}{\sqrt{m}\sqrt{n}}$ since $\sqrt{n}>\sqrt{m}$

$= \frac{2}{\sqrt{m}} \overset{set}{<} \varepsilon$

$\frac{\sqrt{m}}{2} > \frac{1}{\varepsilon} \Rightarrow \sqrt{m} > \frac{2}{\varepsilon}$

$\Rightarrow m > \frac{4}{\varepsilon^2}$

Choose $N = \frac{4}{\varepsilon^2}$

3. $s_n = \left[\cos\left(\frac{n\pi}{3}\right)\right]^n$

$= \left(\left(\frac{1}{2}\right)^1, \left(-\frac{1}{2}\right)^2, (-1)^3, \left(-\frac{1}{2}\right)^4, \left(\frac{1}{2}\right)^5, (1)^6, \left(\frac{1}{2}\right)^7, \ldots \right.$

$= \left(\frac{1}{2}, \frac{1}{4}, -1, \frac{1}{16}, \frac{1}{32}, 1, \frac{1}{128}, \ldots \right.$

(a) $\boxed{\liminf s_n = -1}$ $\boxed{\limsup s_n = 1}$

(b) Set of subsequential limits : $\boxed{\{-1, 0, 1\}}$

4. (a) $\sum \frac{ne^n}{(2n)!}$

Ratio: $\lim_{n\to\infty} \left| \frac{(n+1)e^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{ne^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{e^n}e^1}{\underbrace{(2n+2)!}_{\cancel{(2n)!}\cdot(2n+1)(2n+2)}} \cdot \frac{\cancel{(2n)!}}{n\cancel{e^n}} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)e}{(2n+1)(2n+2)n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(n+1)}e}{(2n+1)\cdot 2\cancel{(n+1)}n} \right|$

$= \lim_{n\to\infty} \left| \frac{e}{2(2n+1)n} \right| = 0 < 1$ $\boxed{\therefore \sum \frac{ne^n}{(2n)!} \text{ converges by the Ratio Test}}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Consider $\int_2^\infty \frac{1}{x\sqrt{\ln x}}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}\,dx$ $\quad$ Bounds: $x \to \infty \Rightarrow u \to \infty$, $x = 2 \Rightarrow u = \ln 2$

$= \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\,du$

$= \int_{\ln 2}^{\infty} u^{-1/2}\,du = 2u^{1/2} \Big|_{\ln 2}^{\infty}$

$= \lim_{u\to\infty} 2\sqrt{u} - 2(\ln 2)^{1/2} = \infty$

The integral diverges $\boxed{\therefore \text{ Series } \sum \frac{1}{n\sqrt{\ln n}} \text{ also diverges}}$

P3

5.(a) Proof
Let s_n be a sequence in $\mathbb{R}$.
Let $\lim s_n = s \in \mathbb{R}$. [Show $\limsup s_n \le s$]

(i) Then $\forall \varepsilon > 0, \exists N$, s.t. $|s_n - s| < \varepsilon \quad \forall n > N$

$\Rightarrow \quad -\varepsilon < s_n - s < \varepsilon \quad \forall n > N$

$s - \varepsilon < s_n < s + \varepsilon \quad \forall n > N$

Thus $s+\varepsilon$ is an upper bound for $\{s_n \mid n > N\}$
therefore $\sup\{s_n \mid n > N\}$ exists by the Completeness Axiom
By definition of supremum
$\sup\{s_n \mid n > N\} \le s + \varepsilon$
Take the limit of both sides:
$$\underbrace{\lim_{N\to\infty} \sup\{s_n \mid n > N\}}_{\limsup s_n} \le \underbrace{\lim_{N\to\infty}(s+\varepsilon)}_{s+\varepsilon}$$
$\limsup s_n \le s + \varepsilon \quad \forall \varepsilon > 0$
By the lemma $\limsup s_n \le s \quad (*)$

(ii) Similarly $s \le \liminf s_n \quad (**)$
By previous theorem $\inf\{s_n \mid n > N\} \le \sup\{s_n \mid n > N\}$
Take the limit as $N \to \infty$
$\Rightarrow \quad \liminf s_n \le \limsup s_n \quad (***)$

Combining $(*)$, $(**)$, $(***)$ gives
$s \le \liminf s_n \le \limsup s_n \le s$
$\therefore$ By the squeeze theorem
$\liminf s_n = \limsup s_n = s \quad \blacksquare$

5(b) Let (x_n) be a sequence of real numbers.
Let $a_n = x_n - x_{n+1} \quad \forall n \in \mathbb{N}$.

(i) Proof Suppose (x_n) converges. [Show $\sum a_n$ converges]
Then $\lim_{n\to\infty} x_n$ exists and is a finite real number, call it x^*.

$\therefore \lim x_n = x^*$

Consider the sequence of partial sums

$$S_k = \sum_{n=1}^{k} a_n$$
$$= \sum_{n=1}^{k} (x_n - x_{n+1})$$
$$= (x_1 - \cancel{x_2}) + (\cancel{x_2} - \cancel{x_3}) + (\cancel{x_3} - \cancel{x_4}) + \dots + (x_{k-1} - \cancel{x_k}) + (\cancel{x_k} - x_{k+1})$$
$$= x_1 - x_{k+1}$$

Take the limit

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} (x_1 - x_{k+1})$$
$$= x_1 - x^* \in \mathbb{R}$$

Since the sequence of partial sums converges, then the series $\sum a_n = \sum (x_n - x_{n+1})$ converges. ∎

(ii) The series $\sum a_n = \sum (x_n - x_{n+1})$ converges to $\boxed{x_1 - x^*}$

Proof P51

5(c)(i) Let Σa_n & Σb be series w/ positive terms.

Let $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = 0$ and let Σa_n converge.

[Show Σb_n converges]

Then $\forall \varepsilon > 0$, $\exists N$ s.t. $\left|\frac{b_n}{a_n} - 0\right| < \varepsilon$

$\Rightarrow \left|\frac{b_n}{a_n}\right| < \varepsilon$

$\Rightarrow \frac{b_n}{a_n} < \varepsilon$ since a_n, b_n are positive.

$\Rightarrow b_n < \varepsilon a_n$

Since Σa_n converges, then so does $\varepsilon \Sigma a_n$ $\forall \varepsilon$, by properties of series.

$\therefore \Sigma b_n$ converges by the Comparison Test.

(ii) $\underbrace{\Sigma \frac{\ln n}{n^3}}_{\Sigma b_n}$ and $\underbrace{\Sigma \frac{1}{n^2}}_{\Sigma a_n}$ which is a converging p-series.

$$\lim \frac{b_n}{a_n} = \lim \frac{\left(\frac{\ln n}{n^3}\right)}{\left(\frac{1}{n^2}\right)} = \lim \frac{\ln n}{n^{\cancel{3}}} \cdot \frac{n^2}{1}$$

$$= \lim \frac{\ln n}{n} = 0$$

$\therefore \Sigma \frac{\ln n}{n^3}$ converges.

6. (12 pts) The Limit Comparison Test can be extended to include a limit of 0 or a limit of ∞.

THE LIMIT COMPARISON TEST EXTENSION

Suppose $\sum a_n$ and $\sum b_n$ are both *series* with *positive* terms.

Briefly explain why you need only consider the case where $\lim a_n = 0$ and $\lim b_n = 0$.

If $\lim a_n \neq 0$ or $\lim b_n \neq 0$, then we would already know that the series $\sum a_n$ & $\sum b_n$ diverge by Test for Divergence

Fill in the blanks with "converges" or "diverges" for the extension below.

(a). If $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ and $\sum a_n$ converges, then $\sum b_n$ converges

$\frac{0}{0}$

[***Briefly*** explain your answer.] Since the indeterminate form $\frac{0}{0}$ results in a 0 limit, that means $b_n \to 0$ faster than $a_n \to 0$. Since $\sum a_n$ converges, then $a_n \to 0$ fast enough, so $b_n \to 0$ faster and $\sum b_n$ would also converge

(b). If $\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$ and $\sum a_n$ diverges, then $\sum b_n$ diverges.

[***Briefly*** explain your answer.]
The indeterminate form $\frac{0}{0}$ results in a limit of ∞, so $b_n \to 0$ slower than $a_n \to 0$. Since $\sum a_n$ diverges, the a_n do not go to 0 fast enough, so $\sum b_n$ also diverges since $b_n \to 0$ even slower.

7. (12 pts) Determine if the following statements are TRUE or FALSE. If the statement is false, give a counterexample or clearly explain why it is not possible.

In this section $s_k = \sum_{n=1}^{k} a_n$ is the k^{th} partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$

T **F** If $\sum |a_n|$ diverges, then $\sum a_n$ is conditionally convergent.

Counterexample: $\sum (-1)^n n$

$\sum |(-1)^n n| = \sum n$ diverges

and $\sum (-1)^n n$ diverges

T F If $\lim_{k\to\infty} s_k$ DNE, then $\sum_{n=1}^{\infty} a_n$ diverges.

T **F** If (s_n) is unbounded above, then $\liminf s_n = \limsup s_n = +\infty$.

Counterexample:

$s_n = (-1)^n n$

$\liminf = -\infty$ $\limsup = +\infty$

T F All convergent sequences are bounded.