Exam 1 Key

1. Triangle Inequality: For $a, b, c \in \mathbb{R}$

(a) $|a-c| \le |a-b| + |b-c|$ or $|a-b| \ge \big||a|-|b|\big|$

(b) A set S is bounded below if $\exists\, m \in \mathbb{R}$, s.t $m \le s$ for all $s \in S$.

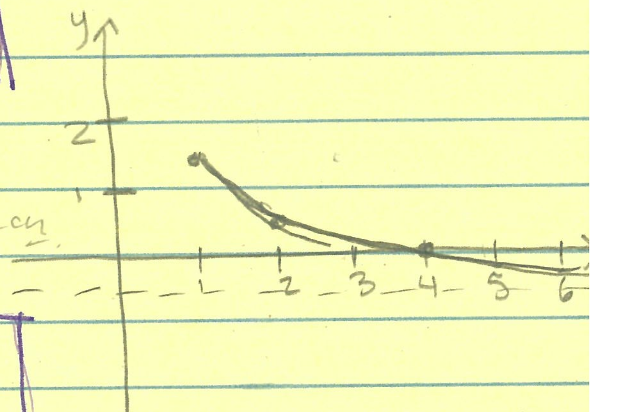
(c) Archimedean Property: If $a>0$ and $b>0$, then $\exists\, n \in \mathbb{N}$ s.t $na > b$.

2(a) $A = \bigcap_{n=1}^{\infty} \left(2+\frac{1}{n}, 5+\frac{1}{n}\right] = (3,6] \cap \left(\frac{5}{2}, \frac{11}{2}\right] \cap \left(\frac{7}{3}, \frac{16}{3}\right] \cap \cdots$

$\rightarrow (3, 5]$

min A: DNE	inf A = 3
max A = 5	sup A = 5

(b) $B = \left\{ \underbrace{\frac{4-x}{2x}}_{y = \frac{2}{x} - \frac{1}{2}} \;\middle|\; x \ge 1, x \in \mathbb{R} \right\}$ Continuous fcn



min B: DNE	inf B = $-\frac{1}{2}$
max B = $\frac{3}{2}$	sup B = $\frac{3}{2}$

3. $S_1 = 3$, $S_{n+1} = \frac{1}{4}(S_n + 1)$ $n \ge 1$

(a)
$S_1 = 3$
$S_2 = \frac{1}{4}(3+1) = 1$
$S_3 = \frac{1}{4}(1+1) = \frac{1}{2}$
$S_4 = \frac{1}{4}\left(\frac{1}{2}+1\right) = \frac{3}{8}$

(b) ie $S_n \to S$ so $S_{n+1} \to S$

$\Rightarrow S = \frac{1}{4}(S+1)$

$\Rightarrow 4S = S+1$

$3S = 1$

$S = \frac{1}{3}$

ie $\boxed{\lim S_n = \frac{1}{3}}$

4 Prove: $\lim \frac{1-2n^2}{3n+4n^2} = -\frac{1}{2}$

Sidework: [Want $\left| \frac{1-2n^2}{3n+4n^2} + \frac{1}{2} \right| < \varepsilon$]

$$|s_n - s| = \left| \frac{1-2n^2}{3n+4n^2} + \frac{1}{2} \right| = \left| \frac{2(1-2n^2)+(3n+4n^2)}{2(3n+4n^2)} \right|$$

$$= \left| \frac{2-4n^2+3n+4n^2}{2(3n+4n^2)} \right|$$

$$= \left| \frac{2+3n}{6n+8n^2} \right| = \frac{2+3n}{6n+8n^2} \quad (*)$$

But $2+3n < 2n+3n = 5n$ and $6n+8n^2 > 8n^2$ $\Big\}$ so $(*)$ $\frac{2n+3}{6n+8n^2} < \frac{5n}{8n^2} = \frac{5}{8n}$

ie

$$(*) < \frac{5}{8n} \overset{\text{want/set}}{<} \varepsilon \Rightarrow \frac{8n}{5} > \frac{1}{\varepsilon} \Rightarrow n > \frac{5}{8\varepsilon}$$

<u>Proof</u>. Let $\varepsilon > 0$ and choose $\underline{N = \frac{5}{8\varepsilon}}$, then for $n > N$ we have

$$|s_n - s| = \left| \frac{1-2n^2}{3n+4n^2} + \frac{1}{2} \right| = \frac{2n+3}{6n+8n^2} < \frac{5}{8n} \text{ by the sidework}$$

$$< \frac{5}{8N} \text{ since } n > N$$

$$= \frac{5}{8(5/8\varepsilon)} = \frac{5}{8} \cdot \frac{8\varepsilon}{5} = \varepsilon$$

ie $|s_n - s| = \left| \frac{1-2n^2}{3n+4n^2} - \left(-\frac{1}{2}\right) \right| < \varepsilon$ whenever $n > N$.

$\therefore \lim \frac{1-2n^2}{3n^2+4n^2} = -\frac{1}{2}$ ■

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5. Prove: If $s_n \to s$ and $t_n \to t$, then $(s_n + t_n) \to s+t$.

<u>Proof</u>. Let $s_n \to s$ and $t_n \to t$ [Show $(s_n+t_n) \to s+t$]

Let $\varepsilon > 0$ [Show $|(s_n+t_n)-(s+t)| < \varepsilon$]

By def. of convergence, for $\varepsilon_1 = \frac{\varepsilon}{2}$, $\exists N_1$ s.t. $|s_n - s| < \frac{\varepsilon}{2}$ whenever $n > N_1$,

and for $\varepsilon_2 = \frac{\varepsilon}{2}$, $\exists N_2$ s.t. $|t_n - t| < \frac{\varepsilon}{2}$ whenever $n > N_2$

Let $N = \max\{N_1, N_2\}$

Consider $|(s_n+t_n)-(s+t)| = |(s_n - s) + (t_n - t)|$

$$\leq |s_n - s| + |t_n - t| \text{ by } \Delta \text{ inequality}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } n > N$$

ie $|(s_n+t_n) - (s+t)| < \varepsilon$ whenever $n > N$ $\therefore$ $s_n + t_n \to s+t$ ■

6. Let a, b, c be elements of a field F. Prove:
If $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Proof. Let $a, b, c \in F$. Let $0 < a < b$. Then since $a \neq 0$ & $b \neq 0$, a^{-1} and b^{-1} exists s.t $aa^{-1} = 1$ & $bb^{-1} = 1$ (by M4)
Also $a^{-1} > 0$ and $b^{-1} > 0$ by CoOFP(vi)
Then Mult. $0 < a < b$ by $a^{-1} > 0$

$\Rightarrow \quad 0 \cdot a^{-1} < a a^{-1} < b a^{-1}$ by O5

$0 < 1 < b a^{-1}$ by CoFP(ii) and M4

Mult. by $\underset{>0}{b^{-1}}$ $\Rightarrow$ $b^{-1} \cdot 0 < b^{-1} \cdot 1 < b^{-1} \cdot b \cdot a^{-1}$ by O5

$0 < b^{-1} < 1 \cdot a^{-1}$ by CoFP(ii) & M4

$0 < b^{-1} < a^{-1}$ by M3

■

7. Proof
Let S be a nonempty bounded subset of $\mathbb{R}$ and let $k \in \mathbb{R}$.
Define $T = \{ks \mid s \in S\}$ Let $k < 0$. [Show sup T = k inf S
Since S is nonempty and bounded so is T.
$\Rightarrow$ sup S, inf S, sup T, inf T all exist.
By definition of infimum,
inf $S \leq s \quad \forall s \in S$
$\Rightarrow k \cdot$ inf $S \geq ks \quad \forall s \in S$ since $k < 0$
$\Rightarrow k \cdot$ inf $S \geq t \quad \forall t \in T$
$k \cdot$ inf S is an upper bound for T
$\Rightarrow$ sup $T \leq k \cdot$ inf S (*)
Also, by def of supremum
$t \leq$ sup $T \quad \forall t \in T$
$\Rightarrow ks \leq$ sup $T \quad \forall s \in S$
$s \geq \frac{1}{k}$ sup T since $k < 0$

$\Rightarrow \frac{1}{k}$ sup T is a lower bound for S
$\Rightarrow \frac{1}{k}$ sup $T \leq$ inf S by def of inf.
$\Rightarrow$ sup $T \geq k \cdot$ inf S since $k<0$ (**)

From (*) & (**)
sup $T = k \cdot$ inf S

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