LIMIT LAWS THEOREM: If $\lim s_n = s, \lim t_n = t$, and $k \in \mathbb{R}$, then

1. $\lim ks_n = k \lim s_n = ks$

2. $\lim(s_n + t_n) = \lim s_n + \lim t_n = s + t$

3. $\lim(s_n \cdot t_n) = \lim s_n \cdot \lim t_n = s \cdot t$

4. $\lim \dfrac{1}{s_n} = \dfrac{1}{\lim s_n} = \dfrac{1}{s}$

5. $\lim \dfrac{t_n}{s_n} = \dfrac{\lim t_n}{\lim s_n} = \dfrac{t}{s}$

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1. $\lim ks_n = k \lim s_n = ks$

PROOF:
Case 1: $k = 0$ This is trivial since $ks_n =$ ____________ $\forall n$. Therefore $\lim ks_n = 0$.

Case 2: $k \neq 0$

Let $\epsilon > 0$. [Show $|ks_n - ks| < \epsilon$ whenever $n > N$.]

Sidework

Show why $\epsilon_1 = \frac{\epsilon}{|k|}$ is a "wise" choice.

Let $\epsilon_1 = \frac{\epsilon}{|k|} > 0$

Since $\lim s_n = s, \exists N$ so that ____________ $<$ ________ $= \frac{\epsilon}{|k|}$ whenever $n > N$.

Then $|ks_n - ks| =$ ______________ $< |k| \cdot \epsilon_1$ whenever $n > N$

But $|k| \cdot \epsilon_1 = |k| \cdot \dfrac{\epsilon}{|k|} = \epsilon$.

i.e. $|ks_n - ks| < \epsilon$ whenever $n > N$.

$\therefore \lim ks_n = ks$ i.e. $\lim ks_n = k \lim s_n$

2. $\lim(s_n + t_n) = \lim s_n + \lim t_n = s + t$

PROOF:
Let $\epsilon > 0$. [Show ______________________ whenever $n > N$.]

Let $\epsilon_1 = \frac{\epsilon}{2}$

Since ______________ , $\exists N_1$ so that $|s_n - s| < \epsilon_1 = \frac{\epsilon}{2}$ whenever $n > N_1$.

Let $\epsilon_2 = \frac{\epsilon}{2}$

Since ______________ , $\exists N_2$ so that $|t_n - t| < \epsilon_2 = \frac{\epsilon}{2}$ whenever ____________ .

Let $N = \max\{N_1, N_2\}$,

then $|s_n + t_n - (s + t)| = |s_n - s + t_n - t| \leq$ __________________ by the triangle inequality.

But $|s_n - s| + |t_n - t| < \epsilon_1 + \epsilon_2 =$ __________ $= \epsilon$ (whenever $n > N$).

i.e. $|s_n + t_n - (s + t)| < \epsilon$ whenever $n > N$.

$\therefore \lim s_n + t_n = s + t = \lim s_n + \lim t_n$

3. $\lim(s_n \cdot t_n) = \lim s_n \cdot \lim t_n = s \cdot t$

Proof

Let $\epsilon > 0$ [Show $|s_n \cdot t_n - st| < \epsilon$.]

Since s_n converges it is ______________ .

Thus $\exists M$ so that $|s_n| \leq M \; \forall n$.

Also $\exists N_1$ so that $|s_n - s| < \epsilon_1 =$ ______________ whenever $n > N_1$.

Since $\lim t_n = t$,

$\exists N_2$ so that $|t_n - t| < \epsilon_2 =$ __________ whenever $n > N_2$.

Let $N =$ __________________, then

$|s_n t_n - st| \leq |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s|$ by sidework

$< |s_n| \cdot$ ________ $+ |t| \cdot$ ________ whenever $n > N$

$< M \cdot \frac{\epsilon}{2M} + |t| \frac{\epsilon}{2|t|+1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

i.e. $|s_n t_n - st| < \epsilon$ whenever $n > N$.

$\therefore \lim s_n \cdot t_n = st = \lim s_n \cdot \lim t_n$. ■

Sidework

$|s_n t_n - st| = |s_n t_n$ __________ $- st|$

$\leq |s_n t_n - s_n t| + |s_n t - st|$

$= |s_n(t_n - t)| + |t(s_n - s)|$

$= |s_n| \cdot |t_n - t| + |t| \cdot |s_n - s|$

$< M \cdot \epsilon_2 + |t| \cdot \epsilon_1$.

Choose ϵ_1 and ϵ_2 (both depend on ϵ),

so that this quantity $< \epsilon$.

$\epsilon_2 = \frac{\epsilon}{2M}$ & $\epsilon_1 = \frac{\epsilon}{2|t|}$ seems to work, but

$|t|$ is a fixed constant that may be 0.

So use $\epsilon_1 = \frac{\epsilon}{2|t|+1} < \frac{\epsilon}{2|t|}$

Then $M \cdot \epsilon_2 + |t| \cdot \epsilon_1$

$= M \cdot \frac{\epsilon}{2M} + |t| \cdot \frac{\epsilon}{2|t|+1}$.

$< \frac{\epsilon}{2} +$ ____________

$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

4. $\lim \frac{1}{s_n} = \frac{1}{\lim s_n} = \frac{1}{s}$

Claim: If $\lim s_n = s$ where $s_n \neq 0\ \forall n$ and $s \neq 0$, then $\inf\{|s_n| \mid n \in \mathbb{N}\} > 0$. **Homework:** Prove this claim.

Proof

Let $\epsilon > 0$ [Show $\left|\frac{1}{s_n} - \frac{1}{s}\right| < \epsilon$]

Let $m = \inf\{|s_n| \mid n \in \mathbb{N}\}$. Then ____________ by the claim.

Then by _________________________ $|s_n| \geq m\ \forall n$.

Let $\epsilon_1 = \epsilon m|s|$

Since s_n converges, $\exists N$ so that $|s_n - s| < \epsilon_1 =$ ___________ whenever $n > N$.

Consider $\left|\frac{1}{s_n} - \frac{1}{s}\right| =$ [Complete the proof from here].

5. $\lim \frac{t_n}{s_n} = \frac{\lim t_n}{\lim s_n} = \frac{t}{s}$ **Homework:** Prove Law 5. [Hint: Use Laws 3 & 4.]

Homework: Finish Worksheet; Section 8, p. 42: #3, 4, 5, 6(a); Section 9, p. 52: #1, 2, 5, 6