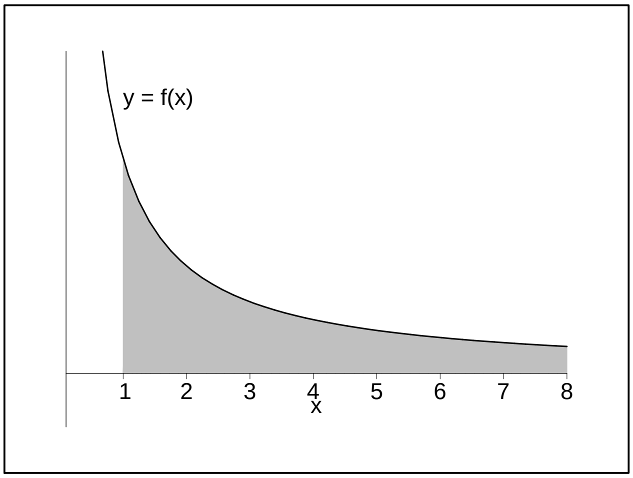
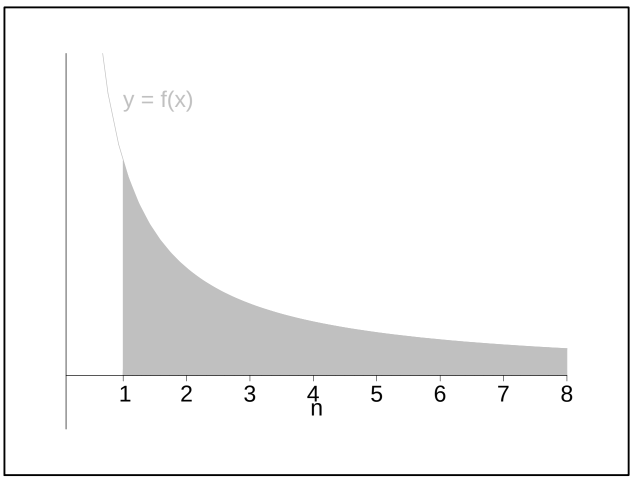
1. Given a continuous, decreasing function $f(x) > 0$, write down an (improper) integral that represents the area under this curve for $x \geq 1$, i.e. on the interval $[1, \infty)$.



2. Let a_n be the sequence such that $f(n) = a_n$. Plot and *<u>label</u>* the points $a_1, a_2, a_3, \ldots$ on the graph below.



3. Estimate the area under the curve using the right endpoint of each subinterval to form rectangles. Sketch and shade these rectangles on the graph. Is this area bigger or smaller than the original shaded area under the curve?

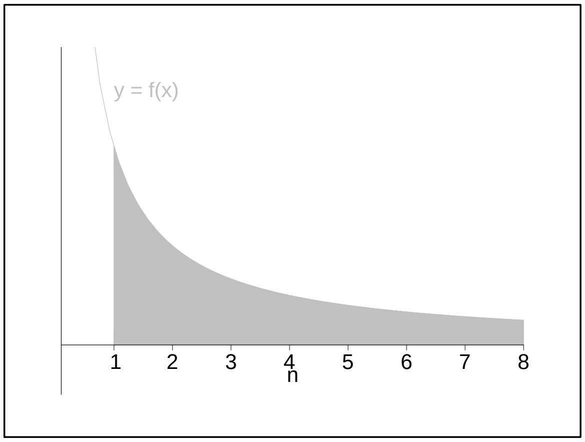
(a). What is the width Δx each rectangle?
What is the height (in terms of the a_n's) of the $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, \ldots$ rectangle?

(b). What is the area of the $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, \ldots$ rectangle?

(c). What is the total area of these (infinitely many) rectangles?

(d). Write this total area in summation form (i.e. as an infinite series including the starting bound).

4. Repeat problem #3 using the left endpoints to form rectangles.



Estimate the area under the curve using the left endpoint of each subinterval to form rectangles. Sketch and shade these rectangles on the graph. Is this area bigger or smaller than the original shaded area under the curve?

(a). What is the width Δx each rectangle?
What is the height (in terms of the a_n's) of the $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, \ldots$ rectangle?

(b). What is the area of the $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, \ldots$ rectangle?

(c). What is the total area of these (infinitely many) rectangles?

(d). Write this total area in summation form (i.e. as an infinite series including the starting bound).

5. In the space below write the expressions you obtained for the 3 areas (#1, 3d, 4d):

Area under curve $f(x) =$

Area using right endpoints =

Area using left endpoints =

Put those areas in order of smallest to largest in the inequality below.

$$0 \leq \qquad\qquad \leq \qquad\qquad \leq$$

Use this inequality to answer the next 2 questions.

6. Suppose the improper integral $\int_1^\infty f(x)\,dx$ converges [i.e it evaluates to a number].

(a). Does $\sum_{n=2}^{\infty} a_n$ converge or diverge? Why?

(b). If $\sum_{n=2}^{\infty} a_n$ converges, does $\sum_{n=1}^{\infty} a_n$ converge? Why?

7. Suppose the improper integral $\int_1^\infty f(x)\,dx$ diverges (to $+\infty$).

(a). Does $\sum_{n=1}^{\infty} a_n$ converge or diverge? Why?

(b). If $\sum_{n=1}^{\infty} a_n$ diverges, does $\sum_{n=2}^{\infty} a_n$ diverge? Why?

The Integral Test

Suppose f is a continuous, positive, and decreasing function on $[1,\infty)$ and let $a_n = f(n)$, then the series $\sum_{n=1}^{\infty} a_n$ converges iff the improper integral $\int_1^\infty f(x)\,dx$ converges.

8. Suppose you are interested in a series that does not start at $n=1$, e.g. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

Do you think that you can still use the integral test?

If so, what integral will you use?

9. Homework: Use the integral test to prove the p-series Test (below). You will need to consider different cases for p.

<u>p**-series Test**</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

10. Homework: Use the <u>Integral Test</u> or <u>p-series Test</u> to determine whether the following series converge or diverge. You may need to review methods of integration.

(a). $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^e}$

(b). $\displaystyle\sum_{n=1}^{\infty} \frac{3}{1+2n}$

(c). $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ [Hint: Integration by parts]

(d). $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

11. Homework: Formally prove the Integral Test.
Hints: The arguments on pages 1-3 are an informal proof – just rewrite them to prove that

$\displaystyle\lim_{k\to\infty} s_k$ converges iff $\displaystyle\int_1^{\infty} f(x)\,dx$ converges.

Recall, from Calc II, that the improper integral is <u>defined</u> as $\displaystyle\lim_{k\to\infty} \int_1^k f(x)\,dx$