Definition

A series $\sum a_n$ satisfies the Cauchy Criterion if its sequence of partial sums s_k is a Cauchy sequence.

i.e. For each $\epsilon > 0$, there exists an N such that $|s_k - s_m| < \epsilon$ whenever $k, m > N$

Fill in the blanks to derive an alternate form of the Cauchy Criterion:

WLOG, assume $k \geq m$.

So, clearly $k > m - 1$ and consider when the Cauchy Criterion holds for k and $m - 1$ both greater than N.

Recall, $s_k =$ ____________, so

$$\begin{aligned} |s_k - s_{m-1}| &= \left|\sum_{n=1}^{k} a_n - \sum_{n=1}^{m-1} a_n\right| \\ &= |(a_1 + a_2 + a_3 + \cdots + a_{m-1} + a_m + a_{m+1} + \cdots + a_k) - (a_1 + a_2 + a_3 + \cdots + a_{m-1})| \\ &= \left|\underline{\hspace{8cm}}\right| \\ &= \left|\sum_{n=m}^{k} a_n\right| \end{aligned}$$

Now substitute this expression into the Cauchy Criterion:

For each $\epsilon > 0$, there exists an N such that $\left|\underline{\hspace{4cm}}\right| < \epsilon$ whenever $k \geq m > N$

Theorem An infinite series $\sum a_n$ converges if and only if it satisfies the Cauchy Criterion.

Proof

$\Rightarrow$: Suppose $\sum a_n$ converges. Then, by definition, the ____________________ converges. Thus, the sequence s_k is Cauchy since __________________________. Therefore, by definition, the series satisfies the Cauchy Criterion. ■

$\Leftarrow$: Suppose the series $\sum a_n$ ______________________. Then, by definition, the sequence of partial sums, s_k, is a Cauchy sequence. Thus, s_k converges and by definition, ____________________. ■

<u>COROLLARY</u> If $\sum a_n$ converges, then $\lim a_n = 0$

PROOF

Let $\epsilon > 0$ and suppose the series $\sum a_n$ converges. Then $\sum a_n$ ______________________________ .

i.e. For $\epsilon > 0$, there exists an N such that $\left|\sum_{n=m}^{k} a_n\right| < \epsilon$ whenever $k \geq m > N$.

In particular, this is true when $k = m$, i.e. _____________ $< \epsilon$ whenever $m > N$. (*)

But $\left|\sum_{n=m}^{m} a_n\right| = |a_m| = |a_m - 0|$, so from (*) we have ________________ whenever $m > N$.

Therefore, $\lim a_m = 0$ ■

Note: m is just the index reference, so this is equivalent to $\lim a_n = 0$

Given all previous definitions and theorems about series, circle all of the following that are true:

Recall $s_k = \sum_{n=1}^{k} a_n$

If $\sum a_n$ diverges, then $\lim a_n \neq 0$

If $\lim a_n = 0$, then $\sum a_n$ converges

If $\lim a_n \neq 0$, then $\sum a_n$ diverges

If $\lim s_k = 23$, then $\lim a_n = 0$

If $\lim a_n = \frac{1}{2}$, then $\sum a_n$ converges

If $\lim s_k = -4$, then we have no information about $\sum a_n$

If $\lim a_n = 0$, then $\lim s_k = s \in \mathbb{R}$

If $\sum a_n = 1$, then s_k converges, but not necessarily to 1

Homework:

Finish Worksheet(s)
Section 14: #5, 6, 9

Find the ***<u>actual sum</u>*** of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ using the partial fraction decomposition $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and considering the sequence of partial sums s_k. [Hint: Write out s_1, s_2, s_3, etc. and find an expression for general s_k – think telescoping.]