DEF A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthogonal set** if each pair of distinct vectors in the set is orthogonal.

i.e. $\mathbf{u}_i \cdot \mathbf{u}_j =$ __0__ for $i \neq j$

NOTE
From properties of the inner product, if $i = j$, then $\mathbf{u}_i \cdot \mathbf{u}_i$ __≥ 0__. Furthermore, $\mathbf{u}_i \cdot \mathbf{u}_i = 0$ iff __$\mathbf{u}_i = \mathbf{0}$__.

1. Verify that the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 3 \end{bmatrix} \right\}$ is an orthogonal set. [Compute three inner products.]

THEOREM If $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent.

PROOF Let S be defined as above. [Show that S is linearly independent.]

Consider the equation [i.e. Show that if $\mathbf{0} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$ then __$c_i = 0$ for $i = 1, \ldots, p$__].

$\mathbf{0} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$ for some scalars $c_1, c_2, \ldots, c_p$. $(*)$

$\mathbf{0} \cdot \mathbf{u}_1 =$ __$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1$__ by taking the inner product with $\mathbf{u}_1$ of both sides.

$0 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_2 \cdot \mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \cdot \mathbf{u}_1$

$0 =$ __$c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2 \cdot 0 + \cdots + c_p \cdot 0$__ since S is an orthogonal set.

$0 = c_1(\mathbf{u}_1 \cdot \mathbf{u}_1)$

$\Rightarrow c_1 = 0$ or __$\mathbf{u}_1 \cdot \mathbf{u}_1 = 0$__.

But if $\mathbf{u}_1 \cdot \mathbf{u}_1 = 0$ then $\mathbf{u}_1 =$ __$\mathbf{0}$__, which is not possible since S is a set of __nonzero (orthogonal) vectors.__

Therefore __$c_1 = 0$__.

Similarly, $c_2 = c_3 = \ldots = c_p = 0$ by taking the dot product of $(*)$ with __$\mathbf{u}_j$ for $j = 2, 3, \ldots, p$__.

i.e. The equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$ only if all of the weights $c_1 = c_2 = \ldots = c_p = 0$.

Therefore (by definition), S is __linearly independent__. ■

COROLLARY If $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is a basis for the subspace W spanned by S.

PROOF Since S spans W and S is linearly independent by ___the last theorem___ , S is is a basis for W by the definition of ___basis___ . ■

2. Let W be the subspace spanned by $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 3 \end{bmatrix} \right\}$.

You showed this was an orthogonal set in #1, so by the corollary it forms a basis for W. Then any vector $\mathbf{y} \in W$ can be written as a linear combination of the vectors in S. i.e. $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ for weights c_1, c_2, and c_3.

Given $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \in W$, find the weights.

DEF An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis of W that is also an orthogonal set.

NOTE The basis in #2 is an orthogonal basis. But why are orthogonal bases nice? [Answered by next theorem.]

THEOREM Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Each $\mathbf{y}$ in W can be written as a linear combination $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$ where the weights are given by

$c_j = \dfrac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$ for $j = 1, 2, \ldots, p$ [NOTE Don't fill in this formula until you have derived it in the proof.]

PROOF Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$.

Let $\mathbf{y}$ be in W. Therefore $\mathbf{y}$ can be written as $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$

Form the dot product with $\mathbf{u}_1$:

$\mathbf{y} \cdot \mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1$

$\mathbf{y} \cdot \mathbf{u}_1 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_2 \cdot \mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \cdot \mathbf{u}_1$

$\mathbf{y} \cdot \mathbf{u}_1 = c_1\mathbf{u}_1 \cdot \mathbf{u}_1$ since it is an ___orthogonal___ set, $\mathbf{u}_i \cdot \mathbf{u}_j =$ ___0___ for $i \neq j$.

Then since $\mathbf{u}_1 \neq \mathbf{0}$ (why? ___linear independence___) $\Rightarrow \mathbf{u}_1 \cdot \mathbf{u}_1$ ___$\neq 0$___ so we can divide by it.

Therefore $c_1 = \dfrac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}$.

Similarly, for the coefficients we obtain $c_j = \dfrac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}$ for $j = 1, 2, \ldots, p$. ■ Write this formula in the theorem above.

3. Redo the problem #2, using this formula to find the weights. [Compare your answers to #2.]

DEF A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthonormal set** if it is an *orthogonal* set of *unit* vectors.

DEF An **orthonormal basis** for a subspace W of $\mathbb{R}^n$ is a basis of W that is also an orthonormal set.

EX The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^n$.

LEMMA A set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is orthonormal if and only if $\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

[NOTE Alternate notation: $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$ where $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ is the Kronecker delta.]

PROOF Do this proof later and turn it in with the 6.2 homework. Go on to the rest of the worksheet now.

[Reminder from last page] Orthonormal $\Rightarrow \mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

Then the theorem on page 2 can be modified for Orthonormal Bases:

<u>THEOREM</u> Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an *orthonormal* basis for a subspace W of $\mathbb{R}^n$. Each $\mathbf{y}$ in W can be written as a linear combination $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$ where the weights are given by

$c_j = \mathbf{y} \cdot \mathbf{u}_j$ for $j = 1, 2, \ldots, p$. [Fill in the correct (simplified) formula.]

4. Given the orthogonal basis from problems #1-3, i.e. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ 3 \end{bmatrix} \right\}$,

(a). Find an orthonormal basis. [Hint: It's already an orthogonal basis. How do you make them of length one?]

(b). Use the formula in the theorem above, to write $\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ as a linear combination of the basis vectors found in part (a). i.e. Find the weights.

Start working on homework until everyone is done with the worksheet.
Homework: Proof on p.3 and Section 6.2, p. 346: #2, 5, 7, 10, 17, 19, 21, 26 // 23(a-d), 25, 27, 28, 29