THEOREM [Section 5.1, Theorem 2]

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are r eigenvectors that correspond to r *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n \times n$ matrix A, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent.

PROOF

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ of an $n \times n$ matrix A.

[Show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent.]

Since $\mathbf{v}_1$ is an ______________ , $\mathbf{v}_1 \neq \mathbf{0}$.

BWOC, suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is ____________________ .

Then by a previous theorem (Sec 1.7), one of the vectors can be written as a ____________________ of the previous vectors.

Let p be the smallest index such that $\mathbf{v}_{p+1}$ is a ____________________ of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent.

Then there exists scalars $c_1, c_2, \ldots, c_p$, such that

$\mathbf{v}_{p+1} =$ ________________________ $(*)$

Multiply $(*)$ by A on the left:

$\Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \cdots + Ac_p\mathbf{v}_p$

$= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_pA\mathbf{v}_p$ by properties of matrix multiplication.

$\lambda_{p+1}\mathbf{v}_{p+1} =$ ______________________________ $(**)$ since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors.

Now, multiply $(*)$ by λ_{p+1}.

$\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} =$ ___________________________________

$= c_1\lambda_{p+1}\mathbf{v}_1 + c_2\lambda_{p+1}\mathbf{v}_2 + \cdots + c_p\lambda_{p+1}\mathbf{v}_p$ $(***)$

Subtract $(**) - (***)$

$\Rightarrow \mathbf{0} = c_1($_____________$)\mathbf{v}_1 + c_2($_____________$)\mathbf{v}_2 + \cdots + c_p($_____________$)\mathbf{v}_p$ $(****)$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent, each coefficient in $(****)$ must be ______ .

i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \ldots, p$.

$\Rightarrow$ __________ or ________________

$\Rightarrow \lambda_k = \lambda_{p+1}$ which is not possible, since the eigenvalues are ____________ .

Therefore, $c_k = 0$ for $k = 1, 2, \ldots, p$.

Substitution into $(*)$ gives $\mathbf{v}_{p+1} =$ ______ $\rightarrow\!\!\!\times$ since it is an ______________ (and must be ___________).

Therefore ____________________________________

THEOREM An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

PROOF

THEOREM Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$, where $p \leq n$.
[Note: $p < n$ means that at least one eigenvalue has multiplicity ________________ .]

(a). The dimension of the eigenspace for an eigenvalue λ_k is less than or equal to the multiplicity of λ_k.

(b). A is diagonalizable if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

PROOF : BYSC

EX: Determine whether A is diagonalizable if A is a 4×4 matrix with eigenvalues $\lambda = -1, 3, -4, -4$, and the basis for each eigenspace, respectively, is

$$\mathcal{B}(\lambda = -1) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{B}(\lambda = 3) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{B}(\lambda = -4) = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Homework: Section 5.3, p. 288: #(2, 3, 5) 7, 9, 13, 16, 19, 21, 24, 25, 27, 29, 31, 32, 35[Use $\lambda = 5, 5, 3, 1, 1$]