[Section 5.1, Theorem 2] If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ are r eigenvectors that correspond to r distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ of an $n \times n$ matrix A_r then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is linearly independent. Photo: Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ be eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ of an $n \times n$ matrix A_r . Since \mathbf{v}_1 is an	More on Diagonalization and Independent	Eigenvectors		Page 1
Phoop Let v ₁ , v ₂ ,, v _r be eigenvectors that correspond to distinct eigenvalues λ ₁ , λ ₂ ,, λ _r of an n × n matrix A. [Show that {v ₁ , v ₂ ,, λ _r of an n × n matrix A. Since v ₁ is an, v ₁ ≠ 0. BWOC, suppose that {v ₁ , v ₂ ,, v _r } is Then by a previous theorem (Sec 1.7), one of the vectors can be written as a of the preceding vectors v ₁ , v ₂ ,, v _p , where v ₁ , v ₂ ,, v _p are linearly independent. Then there exists scalars c ₁ , c ₂ ,, c _p , such that v _{p+1} =	<u>THEOREM</u> If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ are r eigenvectors that correspond then $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is linearly independent.	nd to $r \ \underline{distinct}$ eigenvalues λ_1	[So $\lambda_2, \ldots, \lambda_r$ of a	ection 5.1, Theorem 2] an $n \times n$ matrix A ,
$[\text{Show that } \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\} \text{ is linearly independent.}]}$ Since \mathbf{v}_1 is an, $\mathbf{v}_1 \neq 0$. BWOC, suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is Then by a previous theorem (Sec 1.7), one of the vectors can be written as a of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$. Let p be the smallest index such that \mathbf{v}_{p+1} is a of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent. Then there exists scalars c_1, c_2, \dots, c_p , such that $\mathbf{v}_{p+1} = \ (*)$ Multiply (*) by A on the left: $\Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \dots + Ac_p\mathbf{v}_p$ by properties of matrix multiplication. $\lambda_{p+1}\mathbf{v}_{p+1} = \ (**)$ is since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \ (**)$ is since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \ (**)$ is since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \ (***)$ Subtract $(**) - (***)$ $\Rightarrow 0 = c_1(\)\mathbf{v}_1 + c_2(\)\mathbf{v}_2 + \dots + c_p(\)\mathbf{v}_p$ $(***)$ Subtract $(**) - (***)$ $\Rightarrow \ or \ or = \ or = \ or = \ or = \$	$\frac{P_{ROOF}}{\text{Let } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r} be eigenvectors that correspondent of the second second$	d to distinct eigenvalues λ_1, λ_2	$,\ldots,\lambda_r$ of an n	$\times n$ matrix A.
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BWOC, suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is Then by a previous theorem (Sec 1.7), one of the vectors can be written as a of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent. Then there exists scalars c_1, c_2, \dots, c_p , such that $\mathbf{v}_{p+1} = \underline{\qquad} (*)$ Multiply (*) by A on the left: $\Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \dots + Ac_p\mathbf{v}_p$ by properties of matrix multiplication. $\lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (***)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \Delta_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (***)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \Delta_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (***)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, \mathbf{v}_{p+1} are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \Delta_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (***)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \Delta_{p+1}\mathbf{v}_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (***)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Subtract $(**) - (***)$ $\Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + \dots + c_p(\underline{\qquad})\mathbf{v}_p$ $(****)$ Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in $(****)$ must be $\underline{\qquad}$. i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $\Rightarrow \underline{\qquad} \mathbf{or} \mathbf{v}_k = \lambda_{p+1}$ which is not possible, since the eigenvalues are $\underline{\qquad}$. Therefore, $c_k = 0$ for $k = 1, 2, \dots, p$. Substitution into $(*)$ gives $\mathbf{v}_{p+1} = \underline{\qquad} \rightarrow *$ since it is an $\underline{\qquad}$ (and must be $\underline{\qquad}$).	Since \mathbf{v}_1 is an, $\mathbf{v}_1 \neq 0$.			
Then by a previous theorem (Sec 1.7), one of the vectors can be written as a of the previous vectors. Let p be the smallest index such that \mathbf{v}_{p+1} is a of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent. Then there exists scalars c_1, c_2, \dots, c_p , such that $\mathbf{v}_{p+1} = \underline{\qquad} (*)$ Multiply (*) by A on the left: $\Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \dots + Ac_p\mathbf{v}_p$ $= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \dots + c_pA\mathbf{v}_p$ by properties of matrix multiplication. $\lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**)$ since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}$ are eigenvectors. Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**)$ Subtract (**) - (***) $\Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + \dots + c_p(\underline{\qquad})\mathbf{v}_p$ (***) Subtract (**) - (***) $\Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + \dots + c_p(\underline{\qquad})\mathbf{v}_p$ (****) Subtract (**) = 0 for all $k = 1, 2, \dots, p$. $\Rightarrow \underline{\qquad} 0 \mathbf{r}_{\mathbf{v}_{k+1}} = \mathbf{v}_{k+1}$ which is not possible, since the eigenvalues are $\underline{\qquad}$. Therefore, $c_k = 0$ for $k = 1, 2, \dots, p$.	BWOC, suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is			
Let p be the smallest index such that \mathbf{v}_{p+1} is a of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$, where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent. Then there exists scalars c_1, c_2, \dots, c_p , such that $\mathbf{v}_{p+1} = \$	Then by a previous theorem (Sec 1.7), one of th previous vectors.	e vectors can be written as a		of the
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$\mathbf{v}_{p+1} = \underline{\qquad} (*)$ Multiply (*) by A on the left: $\Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \dots + Ac_p\mathbf{v}_p$ $= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \dots + c_pA\mathbf{v}_p \qquad \text{by properties of matrix multiplication.}$ $\lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**) \text{ since } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1} \text{ are eigenvectors.}$ Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**)$ Subtract $(**) - (***)$ $\Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + \dots + c_p(\underline{\qquad})\mathbf{v}_p \qquad (****)$ Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in $(****)$ must be $\underline{\qquad}$. i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $\Rightarrow \underline{\qquad} \text{or} \qquad \qquad$	Then there exists scalars c_1, c_2, \ldots, c_p , such that			
Multiply (*) by A on the left: $\Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \dots + Ac_p\mathbf{v}_p$ $= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \dots + c_pA\mathbf{v}_p \qquad \text{by properties of matrix multiplication.}$ $\lambda_{p+1}\mathbf{v}_{p+1} = (**) \text{ since } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1} \text{ are eigenvectors.}$ Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = (***) \text{ subtract } (**) - (***)$ Subtract $(**) - (***)$ $\Rightarrow 0 = c_1(())\mathbf{v}_1 + c_2() \mathbf{v}_2 + \dots + c_p\lambda_{p+1}\mathbf{v}_p \qquad (***)$ Subtract $(**) - (***)$ $\Rightarrow 0 = c_1() 0 \mathbf{v}_1 + c_2() \mathbf{v}_2 + \dots + c_p() \mathbf{v}_p \qquad (****)$ Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in $(****)$ must be $(****)$. i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $\Rightarrow (***) \mathbf{v}_k = \lambda_{p+1} \text{ which is not possible, since the eigenvalues are } .$ Therefore, $c_k = 0$ for $k = 1, 2, \dots, p$. Substitution into (*) gives $\mathbf{v}_{p+1} = (**) \mathbf{v}_p$ since it is an } (and must be).	$\mathbf{v}_{p+1} =$	(*)		
$ \Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \dots + Ac_p\mathbf{v}_p $ $ = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \dots + c_pA\mathbf{v}_p $ by properties of matrix multiplication. $ \lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**) \text{ since } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1} \text{ are eigenvectors.} $ Now, multiply (*) by λ_{p+1} . $ \Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} (**) $ Subtract (**) - (***) $ \Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + \dots + c_p(\underline{\qquad})\mathbf{v}_p \qquad (***) $ Subtract (**) - (***) $ \Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + \dots + c_p(\underline{\qquad})\mathbf{v}_p \qquad (****) $ Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in (****) must be $\underline{\qquad}$. i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $ \Rightarrow \underbrace{\qquad} 0 \mathbf{r}_{\qquad \qquad $	Multiply $(*)$ by A on the left:			
$= c_{1}A\mathbf{v}_{1} + c_{2}A\mathbf{v}_{2} + \dots + c_{p}A\mathbf{v}_{p} $ by properties of matrix multiplication. $\lambda_{p+1}\mathbf{v}_{p+1} = (**) \text{ since } \mathbf{v}_{1}, \mathbf{v}_{2}, \dots, \mathbf{v}_{p}, \mathbf{v}_{p+1} \text{ are eigenvectors.}$ Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = (**)$ $= c_{1}\lambda_{p+1}\mathbf{v}_{1} + c_{2}\lambda_{p+1}\mathbf{v}_{2} + \dots + c_{p}\lambda_{p+1}\mathbf{v}_{p} (***)$ Subtract (**) - (***) $\Rightarrow 0 = c_{1}(())\mathbf{v}_{1} + c_{2}()\mathbf{v}_{2} + \dots + c_{p}()\mathbf{v}_{p} (****)$ Since $\mathbf{v}_{1}, \mathbf{v}_{2}, \dots, \mathbf{v}_{p}$ are linearly independent, each coefficient in (****) must be i.e. $c_{k}(\lambda_{k} - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $\Rightarrow \qquad \qquad$	$\Rightarrow A\mathbf{v}_{p+1} = Ac_1\mathbf{v}_1 + Ac_2\mathbf{v}_2 + \dots + Ac_p\mathbf{v}_p$			
$\begin{split} \lambda_{p+1} \mathbf{v}_{p+1} &= \underline{\qquad} (**) \text{ since } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p, \mathbf{v}_{p+1} \text{ are eigenvectors.} \\ \\ \text{Now, multiply } (*) \text{ by } \lambda_{p+1}. \\ &\Rightarrow \lambda_{p+1} \mathbf{v}_{p+1} = \underline{\qquad} \\ &= c_1 \lambda_{p+1} \mathbf{v}_1 + c_2 \lambda_{p+1} \mathbf{v}_2 + \dots + c_p \lambda_{p+1} \mathbf{v}_p \qquad (***) \\ \\ \text{Subtract } (**) - (***) \\ &\Rightarrow 0 = c_1(\underline{\qquad}) \mathbf{v}_1 + c_2(\underline{\qquad}) \mathbf{v}_2 + \dots + c_p(\underline{\qquad}) \mathbf{v}_p \qquad (****) \\ \\ \text{Since } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p \text{ are linearly independent, each coefficient in } (****) \text{ must be } \underline{\qquad}. \\ \\ \text{i.e. } c_k (\lambda_k - \lambda_{p+1}) = 0 \text{ for all } k = 1, 2, \dots, p. \\ \\ \Rightarrow \underline{\qquad} \qquad $	$= c_1 A \mathbf{v}_1 + c_2 A \mathbf{v}_2 + \dots + c_p A \mathbf{v}_p$	by properties of matrix m	ultiplication.	
Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1}\mathbf{v}_{p+1} = \underline{\qquad} \\ = c_1\lambda_{p+1}\mathbf{v}_1 + c_2\lambda_{p+1}\mathbf{v}_2 + \dots + c_p\lambda_{p+1}\mathbf{v}_p \qquad (***)$ Subtract (**) - (***) $\Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + \dots + c_p(\underline{\qquad})\mathbf{v}_p \qquad (****)$ Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in (****) must be $\underline{\qquad}$. i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $\Rightarrow \qquad \qquad$	$\lambda_{p+1}\mathbf{v}_{p+1} = _$	$(**)$ since $\mathbf{v}_1, \mathbf{v}_2,$	$\ldots, \mathbf{v}_p, \mathbf{v}_{p+1}$ ar	e eigenvectors.
Subtract $(**) - (***)$ $\Rightarrow 0 = c_1(\underline{})\mathbf{v}_1 + c_2(\underline{})\mathbf{v}_2 + \dots + c_p(\underline{})\mathbf{v}_p (****)$ Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in $(****)$ must be i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $\Rightarrow \qquad \qquad$	Now, multiply (*) by λ_{p+1} . $\Rightarrow \lambda_{p+1} \mathbf{v}_{p+1} = \underline{\qquad}$ $= c_1 \lambda_{p+1} \mathbf{v}_1 + c_2 \lambda_{p+1} \mathbf{v}_2 + \dots + c_n$	$c_p \lambda_{p+1} \mathbf{v}_p \qquad (***)$		
$\Rightarrow 0 = c_1(\underline{})\mathbf{v}_1 + c_2(\underline{})\mathbf{v}_2 + \dots + c_p(\underline{})\mathbf{v}_p \qquad (****)$ Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in $(****)$ must be i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. $\Rightarrow \qquad \qquad$	Subtract $(**) - (***)$			
Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each coefficient in $(* * **)$ must be i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2, \dots, p$. \Rightarrow or $\Rightarrow \lambda_k = \lambda_{p+1}$ which is not possible, since the eigenvalues are Therefore, $c_k = 0$ for $k = 1, 2, \dots, p$. Substitution into (*) gives $\mathbf{v}_{p+1} = \underline{\qquad} \rightarrow $ since it is an (and must be). Therefore	$\Rightarrow 0 = c_1(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + c_2(\underline{\qquad})\mathbf{v}_1 + c_2(\underline{\qquad})\mathbf{v}_2 + c$	$\mathbf{v}_2 + \cdots + c_p($	\mathbf{v}_p	(* * **)
i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2,, p$. $\Rightarrow $	Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are linearly independent, each	h coefficient in $(* * **)$ must b	e	
$\Rightarrow \ \text{ or } \ $ $\Rightarrow \lambda_k = \lambda_{p+1} \text{ which is not possible, since the eigenvalues are } _ \$ Therefore, $c_k = 0$ for $k = 1, 2, \dots, p$. Substitution into (*) gives $\mathbf{v}_{p+1} = _$ \rightarrow since it is an (and must be). Therefore	i.e. $c_k(\lambda_k - \lambda_{p+1}) = 0$ for all $k = 1, 2,, p$.			
$\Rightarrow \lambda_k = \lambda_{p+1} \text{ which is not possible, since the eigenvalues are } .$ Therefore, $c_k = 0$ for $k = 1, 2, \dots, p$. Substitution into (*) gives $\mathbf{v}_{p+1} = \underline{\qquad} \rightarrow -$ since it is an (and must be). Therefore	⇒ or			
Therefore, $c_k = 0$ for $k = 1, 2,, p$. Substitution into (*) gives $\mathbf{v}_{p+1} = \underline{\qquad} \rightarrow $ since it is an (and must be). Therefore	$\Rightarrow \lambda_k = \lambda_{p+1}$ which	ch is not possible, since the eig	envalues are	·
Substitution into (*) gives $\mathbf{v}_{p+1} = \underline{\qquad} \rightarrow $ since it is an (and must be). Therefore	Therefore, $c_k = 0$ for $k = 1, 2,, p$.			
Therefore	Substitution into (*) gives $\mathbf{v}_{p+1} = \underline{\qquad} \rightarrow$	- since it is an	(and must b	e).
	Therefore			

<u>THEOREM</u> An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof

<u>THEOREM</u> Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_p$, where $p \leq n$. [Note: p < n means that at least one eigenvalue has multiplicity ______.]

(a). The dimension of the eigenspace for an eigenvalue λ_k is less than or equal to the multiplicity of λ_k .

(b). A is diagonalizable if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k .

 \underline{PROOF} : BYSC

EX: Determine whether A is diagonalizable if A is a 4×4 matrix with eigenvalues $\lambda = -1, 3, -4, -4$, and the basis for each eigenspace, respectively, is

$$\mathcal{B}(\lambda = -1) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \qquad \mathcal{B}(\lambda = 3) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \qquad \mathcal{B}(\lambda = -4) = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Homework: Section 5.3, p. 288: #(2, 3, 5) 7, 9, 13, 16, 19, 21, 24, 25, 27, 29, 31, 32, 35[Use $\lambda = 5, 5, 3, 1, 1$]