1. Given $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

(a). For $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, compute $A\mathbf{u}$. Sketch $\mathbf{u}$ and the resulting image $A\mathbf{u}$ on the same set of axes.

(b). For $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, compute $A\mathbf{v}$. Sketch $\mathbf{v}$ and the resulting image $A\mathbf{v}$ on the same set of axes.

(c). What, if anything, do you notice special about either of these cases?

<u>Def</u> Let A be an ____________ matrix.

- An ________________ of A is a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution $\mathbf{x}$.
- An _________________ of A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ (the eigenvalue).

<u>Note</u>: We say $\mathbf{x}$ is the eigenvector associated with λ.

In the above example, $\lambda = -2$ is the eigenvector with eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

2. Show that $\mathbf{w} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ is an eigenvector of A and determine the eigenvalue.

Question (rhetorical): Are $\lambda = -2, 3$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ the only eigenvalues and eigenvectors of A?
[Let's see on the next page.]

Eigenvectors are nonzero solutions to $A\mathbf{x} = \lambda\mathbf{x}$

$\Rightarrow$ By subtracting: __________________

$\Rightarrow$ Factor: ____________________ where I is the ____________ identity matrix.

$\Rightarrow$ By the IMT, this equation will have a nontrivial solution if the matrix $A - \lambda I$ is ___________________ .

$\Rightarrow$ $\det(A - \lambda I) =$ ________ (also by the IMT).

<u>Steps for finding eigenvalues</u>:

(1). Find and simplify the matrix $A - \lambda I$.

(2). Compute $\det(A - \lambda I)$

(3). Set $\det(A - \lambda I) = 0$ and solve for λ.

3. Find the eigenvalues $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$

(1). $A - \lambda I = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} -$

(2). $|A - \lambda I| = \left| \qquad\qquad \right| =$

(3).

So the ___________ eigenvalues are $\lambda =$ ____________.

<u>Def</u> The scalar equation $\det(A - \lambda I) = 0$ is called the ____________________________ .

<u>Theorem</u> A scalar λ is an eigenvalue of an $n \times n$ matrix A iff λ satisfies the characteristic equation $\det(A - \lambda I) = 0$.

<u>Note</u>: For an $n \times n$ matrix A, the characteristic equation is an __________ -order polynomial (___________) and has exactly n roots if you count repeated roots and complex roots.

Def The multiplicity of an eigenvalue λ is the number of times λ is a root of the characteristic equation.

4. Find the eigenvalues $A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and state their multiplicity.

5. Based on the last problem, complete the following theorem about triangular matrices.

Theorem The eigenvalues of an $n \times n$ triangular matrix are ______________________________ .

(Proof – to be done as homework)

QUESTION: Now that we know how to find the eigenvalues, how do we find the associated eigenvectors?

Eigenvectors are nonzero solutions to $A\mathbf{x} = \lambda\mathbf{x}$, which is equivalent to ____________________ .

$\Rightarrow$ Find the nontrivial solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$

<u>Steps for finding eigenvectors</u>:

(1). Find and simplify $A - \lambda I$

(2). Use row reduction to find the nontrivial solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

6. Back to the first matrix $A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$, find the eigenvector(s) associated with $\lambda = 3$.

(1). $A - 3I =$

(2).

7. Find the eigenvector(s) associated with $\lambda = -2$.

8. Did you get the same eigenvectors from before?

9. Are the eigenvectors unique? Why or why not?

Homework: Explain in your own words what the Null Space of a matrix A is.
Proof on p.3
Section 5.1, p. 273: #1, 4, 7, [11, 15, 16 Note: Eigenspace] , 17, 23, 24
Section 5.2, p. 281: #3, 8, 9, 13, 20