<u>THEOREM</u> Cramer's Rule

Let A be an $n \times n$ invertible matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ for the equation $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A} = \frac{|A_i(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of matrix A with the vector $\mathbf{b}$.

i.e. $A_i(\mathbf{b}) = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \ldots \;\; \mathbf{b} \;\; \ldots \;\; \mathbf{a}_n]$

$\uparrow$ column i

<u>PROOF</u> Let A be ____________________ and let $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{b}$ in $\mathbb{R}^n$.

Claim 1: $A_i(\mathbf{b})$ can be written as the product $AI_i(\mathbf{x})$

(i.e. $A_i(\mathbf{b}) = AI_i(\mathbf{x})$ where $I_i(\mathbf{x})$ is the $n \times n$ identity matrix with the i^{th} column replaced by vector $\mathbf{x}$.)

Proof (of Claim 1):

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ denote the ________________ and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the ________________ .

Then

$$\begin{aligned}
AI_i(\mathbf{x}) &= A[\qquad\qquad\qquad\qquad] \\
&= [A\mathbf{e}_1 \;\; A\mathbf{e}_2 \;\; \ldots \;\; A\mathbf{x} \;\; \ldots \;\; A\mathbf{e}_n] \\
&= [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \ldots \;\; \mathbf{b} \;\; \ldots \;\; \mathbf{a}_n] \\
&= A_i(\mathbf{b}) \qquad\qquad \blacksquare\text{(Claim 1)}
\end{aligned}$$

Claim 2: $\det I_i(\mathbf{x}) = x_i$

Proof (of Claim 2): Consider the $n \times n$ matrix $I_i(\mathbf{x}) = \begin{bmatrix} 1 & 0 & \ldots & x_1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & x_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & x_i & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \ddots & & \vdots \\ 0 & 0 & \ldots & x_n & 0 & \ldots & 1 \end{bmatrix}$

and compute the determinant using a cofactor expansion along row i:

$$\begin{aligned}
\det I_i(\mathbf{x}) &= \underline{\qquad\qquad} \cdot \det(I_{n-1}) && \text{since all the other elements in the row are zero.} \\
&= x_i \cdot 1 \cdot 1 && \text{since the determinant of any size identity matrix is 1.} \\
&= x_i \quad \blacksquare\text{(Claim 2)}
\end{aligned}$$

over $\longrightarrow$

Back to Main Proof:

From Claim 1, we have

$$\rule{2cm}{0.4pt} = \rule{2cm}{0.4pt}$$

$$\Rightarrow \det A_i(\mathbf{b}) = \det AI_i(\mathbf{x}) \quad \text{by taking the determinant of both sides.}$$

$$\det A_i(\mathbf{b}) = \rule{3.5cm}{0.4pt} \quad \text{by previous theorem}$$

$$\det A_i(\mathbf{b}) = \det A \cdot x_i \quad \text{by Claim 2}$$

Then since ______________________________ we can divide to get

$$\Rightarrow x_i = \frac{\det A_i(\mathbf{b})}{\det A} \qquad \blacksquare$$

Ex: Use Cramer's Rule to solve
$$\begin{aligned} -2x_1 + 4x_2 &= 7 \\ 5x_1 + 3x_2 &= 2 \end{aligned}$$

Note:

Application of Cramer's Rule to Differential Equations

The **LaPlace Transform** $\mathcal{L}\{f(t)\} = \int_0^\infty e^{st} f(t)\, dt =$

converts a (system of) linear differential equation(s) into a (system of) linear ____________ equation(s).

Ex Applying the LaPlace Transform to the system of differential equations [Details BYSC]

$$\begin{aligned} \frac{dx}{dt} &= 2x + y \qquad x(0) = 1 \\ \frac{dy}{dt} &= 3x + 4y \qquad y(0) = 0 \end{aligned}$$

results in the following algebraic system

$$\begin{aligned} (s-2)x_1 &- \quad x_2 &= 1 \\ -3x_1 &+ (s-4)x_2 &= 0 \end{aligned}$$

(a). Determine the value(s) of s for which the system has a unique solution.

(b). Use Cramer's Rule to find the solution.

Note: The solution to the original differential equation is found by determining the inverse LaPlace transform of x_1 and x_2 (BYSC). But the answer is $x(t) = L^{-1}\left\{\frac{s-4}{(s-1)(s-5)}\right\} = \frac{3}{4}e^t + \frac{1}{4}e^{5t}$ and $y(t) = L^{-1}\left\{\frac{3}{(s-1)(s-5)}\right\} = -\frac{3}{4}e^t + \frac{3}{4}e^{5t}$

Recall the formula for finding the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $A^{-1} =$

Is there a similar formula for $n \times n$ matrices?

<u>DEF</u> Let A be and $n \times n$ matrix. The ______________________________ , denoted adj A is the ____________ given by

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & c_{32} & \cdots & c_{n2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix}$$

where $c_{ij} =$

Note: To find the adjoint of A, replace each entry in A with ________________ , then _______________ it.

i.e. adj $A =$

<u>THEOREM</u> If A is an $n \times n$ invertible matrix, then $A^{-1} = \dfrac{1}{\det A} \text{adj } A$.

<u>PROOF</u> [See book. It comes from applying Cramer's Rule.]

<u>EX</u>: Verify that the above theorem gives the formula for the inverse of a 2×2 matrix.

[On Board Example]

Homework (slightly different than assignment sheet): Section 3.3, p. 185: #2, 5, 7, 9, 13, 16,