THEOREM A square $n \times n$ matrix A is invertible if and only if $\det A \neq 0$

Before we can prove this, we need more theorems.

THEOREM Let E be an $n \times n$ elementary matrix. Then $\det E = \begin{cases} 1, & \text{if } E \text{ is a row replacement} \\ -1, & \text{if } E \text{ is a row interchange} \\ k, & \text{if } E \text{ is a row scaled by } k \end{cases}$

PROOF Let E be an $n \times n$ elementary matrix.

Basis (n=2): $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a). For row replacement, there are only 2 possible elementary matrices:

$kR_2 + R_1 \to R_1 : E_1 = \begin{bmatrix} & \\ & \end{bmatrix}$ and $kR_1 + R_2 \to R_2 : E_2 = \begin{bmatrix} & \\ & \end{bmatrix}$

Then $\det E_1 =$ ________________________ and $\det E_2 = (1)(1) - (0)(k) = 1$ $\surd$

(b). For row interchange, the only elementary matrix is $E = \begin{bmatrix} & \\ & \end{bmatrix}$

Then $\det E_1 =$ ________________________ $\surd$

(c). For row scaling by k, there are only 2 possible elementary matrices:

$E_1 = \begin{bmatrix} & \\ & \end{bmatrix}$ and $E_2 = \begin{bmatrix} & \\ & \end{bmatrix}$

Then $\det E_1 =$ ________________________ and $\det E_2 =$ ________________________ $\surd$

Induction:

Assume true for $n = k$.

ie. $\det E = \begin{cases} 1, & \text{if } E \text{ is a row replacement} \\ -1, & \text{if } E \text{ is a row interchange} \\ k, & \text{if } E \text{ is a row scaled by } k \end{cases}$ ________________________ . [Show true for an $(k+1) \times (k+1)$ matrix.]

Since elementary row operations only affect 1 or 2 rows, compute the determinant by a cofactor expansion along ________________________ , call it row i.

This row is given by $[0 \;\; 0 \;\; \ldots \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0]$, where the 1 is in the i^{th} position.

So by cofactor expansion,

$$\begin{aligned} \det E &= \qquad\qquad\qquad \text{where } E_{ii} \text{ is the } \_\_\_\_\_\_\_\_ \text{ submatrix with row } i \text{ and column } i \text{ deleted.} \\ &= (-1)^{2i} \det(E_{ii}) \\ &= \det(E_{ii}) \\ &= \begin{cases} 1, & \text{if } E_{ii} \text{ is a row replacement} \\ -1, & \text{if } E_{ii} \text{ is a row interchange} \\ k, & \text{if } E_{ii} \text{ is a row scaled by } k \end{cases} \quad \text{by } \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{aligned}$$

i.e., it is true for $n = k + 1$. Therefore, by mathematical induction it is true for all $n \times n$ elementary matrices. ■

<u>THEOREM</u> If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EA) = \det(E)\det(A) = \begin{cases} \det(A), & \text{if row replacement} \\ -\det(A), & \text{if row interchange} \\ k\det(A), & \text{if row scaled by } k \end{cases}$$

Note: This is Theorem 3 on p. 169, with B written as EA. See book for proof.

<u>EX</u>: $A = \begin{bmatrix} 2 & 4 \\ 1 & -5 \end{bmatrix}$ $\det A =$

$-2R_2 + R_1 \to R_1$: $B_1 =$

$R_2 \leftrightarrow R_1$: $B_2 =$

$-3R_2 \to R_2$: $B_3 =$

<u>THEOREM</u> If A is an $n \times n$ matrix that is reduced to echelon form U using ***only*** row interchanges and row replacements. Then

$$\begin{aligned} \det A &= (-1)^r \det U \qquad \text{where } r \text{ is the number of row interchanges used.} \\ &= (-1)^r \qquad \text{(i.e. \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)} \end{aligned}$$

since the echelon form is always upper triangular.

<u>EX</u>: Reduce the following matrix to echelon form and compute the determinant. Compare with computing the determinant by cofactor expansion. $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & -2 \\ 0 & 1 & 2 \end{bmatrix}$